MINIMAX SAMPLING WITH ARBITRARY SPACES

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ABSTRACT

We consider non-ideal sampling and reconstruction schemes in which the sampling and reconstruction spaces as well as the input signal can be arbitrary. To obtain a good reconstruction of the signal in the reconstruction space from arbitrary samples, we suggest processing the samples prior to reconstruction with a linear transformation that is designed to minimize the worst-case squared-norm error between the reconstructed signal, and the best possible (but usually unattainable) approximation of the signal in the reconstruction space. We show both theoretically and through a simulation that if the input signal does not lie in the reconstruction space, then this method can outperform the consistent reconstruction method previously proposed for this problem.

1. INTRODUCTION

The most common setting considered in the sampling literature is that introduced by Shannon's sampling theorem, in which the input is assumed to be bandlimited, the samples of the signals are ideal, *i.e.,* they are equal to the signal values at a set of sampling points, and the reconstructed signal is also a bandlimited function, created by using the $sinc$ interpolation kernel. In practice, however, the input signal is never perfectly bandlimited, and the sampling is never ideal. Another drawback of the Shannon paradigm is the difficulty in implementing the infinite $sinc$ interpolating kernel.

To overcome these limitations of the traditional sampling framework, a more recent approach to sampling is to consider samples that can be represented as the inner products of the input signal x with a set of sampling vectors, which form a possibly overcomplete basis (frame) for the sampling space, denoted by $\mathcal{S}$. The problem then is to reconstruct x from these samples, using a set of vectors that form an overcomplete bases for a subspace $\mathcal{W}$, referred to as the reconstruction space. The input signal x is assumed to lie in an arbitrary Hilbert space $\mathcal{H}$. Since, in this framework, the reconstructed signal is constrained to lie in $\mathcal{W}$, if x does not lie in $\mathcal{W}$, then perfect reconstruction is not possible, regardless of the sampling and reconstruction method.

If $\mathcal{S} = \mathcal{W}$, then by properly processing the samples prior to reconstruction we can obtain the minimal-norm approximation to x in the space $\mathcal{W}$, given by $P_{\mathcal{W}}x$, where $P_{\mathcal{A}}$ denotes the orthogonal projection onto the subspace $\mathcal{A}$. However, if $\mathcal{S}$ is arbitrary, then the minimal norm approximation is usually unattainable. Therefore, our problem is to design a linear transformation H to process the given samples prior to reconstruction (see Fig.1), so that the reconstructed signal $\hat{x}$ is close to x in some sense.

The problem depicted in Fig. 1 was first considered in [1], for the case in which the sampling and reconstruction spaces are shift-invariant (SI) subspaces of L_2, *i.e.,* spaces generated by translates of an appropriately chosen function. The transformation H was chosen such that $\hat{x}$ is a *consistent reconstruction* of x, namely a reconstruction with the property that is yields the same samples as x. This approach was then generalized in [2, 3] to arbitrary sampling and reconstruction spaces, as well as arbitrary input spaces $\mathcal{H}$. The resulting reconstruction is $\hat{x} = E_{\mathcal{W}\mathcal{S}^{\perp}}x$ where $E_{\mathcal{W}\mathcal{S}^{\perp}}$ is the oblique projection onto $\mathcal{W}$ along $\mathcal{S}^{\perp}$, and $\mathcal{S}^{\perp}$ is the orthogonal complement of $\mathcal{S}$. Note, however, that the fact that x and $\hat{x}$ yield the same samples does not necessarily imply that $\hat{x}$ is close to x.

To obtain a reconstruction $\hat{x}$ that is close to x, we may choose H to minimize the error norm $\|\hat{x} - x\|^2$. However, since the error norm depends on x, which is unknown, it cannot generally be minimized directly. Instead, we may consider a worst-case approach in which we design H to minimize the worst-case error norm over all bounded energy inputs x. The drawback of such a minimax approach is that it may tend to be overconservative, since it considers the worst possible input choice. Indeed, as we will show in Section 2, the minimax transformation is given by $H = 0$. To obtain a reconstruction that is less pessimistic, we consider a competitive approach, similar in spirit to the approaches in [4, 5], in which we seek the H that minimizes the worst-case *regret* instead of the worst-case error norm, where we define the regret as the difference between the error norm achieved by a transformation H, and the smallest possible error norm achievable when $\mathcal{S} = \mathcal{W}$. As we will show in Section 2, the resulting reconstruction is given by $\hat{x} = P_{\mathcal{W}}P_{\mathcal{S}}x$. We then show, both through analyzing the reconstruction error in Section 3, and through simulation in Section 4, that the minimax regret approach can improve the performance over the traditional consistent reconstruction approach.

2. MINIMAX REGRET ESTIMATOR

Consider the sampling scheme depicted in Fig.1, in which we are given a set of samples $c[n] = \langle s_n, x\rangle$ of a signal $x \in \mathcal{H}$, where the vectors $\{s_n\}$ are the sampling vectors and form a frame (*i.e.,* an overcomplete basis) for the sampling space $\mathcal{S}$, and $\langle x, y\rangle$ is the inner product on $\mathcal{H}$. The reconstruction $\hat{x}$ of x is obtained as $\hat{x} = \sum_n d[n]w_n$ for a set of samples $d[n]$, where $\{w_n\}$ are the reconstruction vectors that form a frame for the reconstruction space $\mathcal{W}$, and $d = Hc$ for some linear transformation H. Here d and c are the sequences in l_2 with elements $d[n]$ and $c[n]$, respectively. Using set transformations[1] we can express c and $\hat{x}$ as $c = S^*x$ and $\hat{x} = Wd = WHS^*x$, where $S, W : l_2 \rightarrow \mathcal{H}$ are the set transformations corresponding to $\{s_n\}$ and $\{w_n\}$, respectively.

[1]A set transformation $V : l_2 \rightarrow \mathcal{H}$ corresponding to vectors $\{v_n\} \in \mathcal{H}$ is defined by $Va = \sum_n a[n]v_n$ for all $a \in l_2$. From the definition of the adjoint, if $a = V^*y$ then $a[n] = \langle v_n, y\rangle$.

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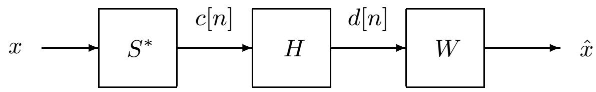

Figure 1: General sampling and reconstruction scheme.

Our problem is to choose H such that $\hat{x}$ is a good approximation of x in some sense. One approach to choosing H is to require that $\hat{x}$ is a consistent reconstruction of x *i.e.,* a reconstruction satisfying $S^*x = S^*\hat{x}$, for all $x \in \mathcal{H}$ [1, 2, 3]. Under this criterion H is given by $H = (S^*W)^\dagger$, where $(\cdot)^\dagger$ denotes the Moore-Penrose pseudo inverse, and $\hat{x}$ is the oblique projection $\hat{x} = E_{\mathcal{W}\mathcal{S}^\perp}x$. If $x \in \mathcal{W}$, then for this choice of H, $\hat{x} = x$. However, if $x \notin \mathcal{W}$, then $\hat{x}$ may not necessarily be close to x.

To obtain a reconstruction $\hat{x}$ that is close to x for $x \notin \mathcal{W}$, we may instead seek the transformation H that minimizes the norm of the reconstruction error $\|x - \hat{x}\|^2$. However, since this norm depends explicitly on x, it cannot in general be minimized directly. Instead, we may consider minimizing the worst possible error norm, over all bounded inputs x, so that H is the solution to

$$\min_H \max_{\|x\| \le L} \|x - \hat{x}\|^2 = \min_H \max_{\|x\| \le L} \|x - WHS^*x\|^2 . \quad (1)$$

Unfortunately, as we now show, the criterion (1) is over-pessimistic and yields the trivial solution $H = 0$. To this end we first note that exchanging the order of the minimum and maximum, $\min_H \max_{\|x\| \le L} \|x - WHS^*x\|^2 \ge \max_{\|x\| \le L} \min_H \|x - WHS^*x\|^2$. To determine the $\widehat{H}$ minimizing $\|x - WHS^*x\|^2$, we note that if $x \in \mathcal{S}^\perp$, then $\|x - WHS^*x\|^2 = \|x\|^2$ for any choice of H. If $x \notin \mathcal{S}^\perp$, then since $\min_{y \in \mathcal{W}} \|x - y\|^2 = \|P_{\mathcal{W}^\perp}x\|^2$, it follows that if we can find an $\widehat{H}$ (depending on x) such that

$$\left\|x - W\widehat{H}S^*x\right\|^2 = \|P_{\mathcal{W}^\perp}x\|^2, \quad (2)$$

then $\widehat{H} = \arg\min_H \|x - WHS^*x\|^2$. Expressing $P_\mathcal{W}$ as $P_\mathcal{W} = W(W^*W)^\dagger W^*$, it is easy to see that with

$$\widehat{H} = \frac{1}{x^*SS^*x}(W^*W)^\dagger W^*xx^*S, \quad (3)$$

(2) is satisfied. Since $\max_{\|x\| \le L, x \notin \mathcal{S}^\perp} \|P_{\mathcal{W}^\perp}x\|^2 = L^2$ and $\max_{\|x\| \le L, x \in \mathcal{S}^\perp} \|x\|^2 = L^2$, we conclude that $\min_H \max_{\|x\| \le L} \|x - WHS^*x\|^2 \ge L^2$. For $H = 0$, $\max_{\|x\| \le L} \|x - WHS^*x\|^2 = \max_{\|x\| \le L} \|x\|^2 = L^2$, and therefore $H = 0$ is the solution to (1).

We note that if $\mathcal{S} = \mathcal{W}$, then following similar arguments we can show that for all $x \in \mathcal{H}$,

$$\min_H \|x - \hat{x}\|^2 = \min_H \|x - WHW^*x\|^2 = \|P_\mathcal{W}x\|^2 , \quad (4)$$

and the minimizing H is $(W^*W)^\dagger$. Thus, in this case, the minimal error can be achieved even without knowledge of x (in contrast with the case in which $\mathcal{S} \neq \mathcal{W}$ in which the optimal H given by (3) depends on x). Therefore, instead of trying to minimize the worst-case error norm $\|\hat{x} - x\|^2$, we consider minimizing the worse-case *regret*, *i.e.,* the worst-case difference between the error obtained in the case in which $\mathcal{S} \neq \mathcal{W}$, and the best possible error attainable when $\mathcal{S} = \mathcal{W}$. Thus, we now consider the problem

$$\begin{aligned} &\min_H \max_{\|x\| \le L} \left(\|x - \hat{x}\|^2 - \|P_{\mathcal{W}^\perp}x\|^2\right) \\ &\quad = \min_H \max_{\|x\| \le L} \|\hat{x} - P_\mathcal{W}x\|^2 . \end{aligned} \quad (5)$$

The solution to (5) is given in the following theorem.

Theorem 1. *Let* H_{REG} *be the solution to*

$$\min_H \max_{\|x\| \le L} \|\hat{x} - P_\mathcal{W}x\|^2 = \min_H \max_{\|x\| \le L} \|WHS^*x - P_\mathcal{W}x\|^2 .$$

Then H_{REG} *is independent of L and is given by*

$$H_{\mathrm{REG}} = (W^*W)^\dagger W^*S(S^*S)^\dagger \quad (6)$$

and $\hat{x} = P_\mathcal{W}P_\mathcal{S}x$.

Proof. We first note that $\max_{\|x\| \le L} \|\hat{x} - P_\mathcal{W}x\|^2 \ge \max_{\|x\| \le L, x \in \mathcal{S}^\perp} \|\hat{x} - P_\mathcal{W}x\|^2$. But since for $x \in \mathcal{S}^\perp$ we have that $\hat{x} = 0$,

$$\begin{aligned} \min_H \max_{\|x\| \le L} \|P_\mathcal{W}x - \hat{x}\|^2 &\ge \max_{\|x\| \le L, x \in \mathcal{S}^\perp} \|P_\mathcal{W}x\|^2 \\ &\ge \max_{\|x\| \le L} \|P_\mathcal{W}P_{\mathcal{S}^\perp}x\|^2 . \end{aligned} \quad (7)$$

The theorem then follows from showing that with $H = H_{\mathrm{REG}}$, the lower bound in (7) is achieved. □

The consistent reconstruction of [1, 2, 3] assumes explicitly that $\mathcal{W}$ and $\mathcal{S}$ satisfy $\mathcal{H} = \mathcal{W} \oplus \mathcal{S}^\perp$, where $\oplus$ denotes the direct sum; otherwise W^*S is not necessarily invertible (or pseudo invertible). In contrast, the reconstruction algorithm of Theorem 1 does not require such an assumption and therefore can be applied to arbitrary subspaces $\mathcal{W}$ and $\mathcal{S}$ of an arbitrary Hilbert space $\mathcal{H}$.

2.1. Reconstruction In Shift-Invariant Spaces

A special case of Theorem 1, first considered in [1], is the case in which $x = x(t)$ is a real signal in L_2, and $\mathcal{S}$ and $\mathcal{W}$ are real SI spaces generated by $\mathcal{W} = \{z(t) = \sum_n a[n]w(t-n)\}$ and $\mathcal{S} = \{f(t) = \sum_n b[n]s(t-n)\}$, where $s(t)$ and $w(t)$ are appropriately chosen functions. In this case, the samples $c[n] = \int s(t-n)x(t)dt$ correspond to samples at times $t = n$ of the output of a filter with impulse response $s(-t)$ with $x(t)$ as its input. The reconstructed signal corresponds to the output of a filter with impulse response $w(t)$, with an impulse train whose values are the corrected measurements $d[n]$ as its input where, from Theorem 1, $d = (W^*W)^\dagger W^*S(S^*S)^\dagger c$ and S and W are the set transformations corresponding to $\{s_n(t) = s(t-n)\}$ and $\{w_n(t) = w(t-n)\}$ respectively. Since $\langle w_i(t), s_n(t)\rangle = g(n-i)$, where $g(t) = w(t) * s(-t)$, W^*S is an infinite Toeplitz matrix, and is therefore equivalent to a filtering operation. Likewise, W^*W and S^*S are equivalent to filtering operations, so that the corrected samples d can be obtained from c using a discrete-time filter with frequency response

$$H(\omega) = \begin{cases} \frac{\sum_{k=-\infty}^{\infty} S(\omega+2\pi k)W^*(\omega+2\pi k)}{\sum_{k=-\infty}^{\infty}|S(\omega+2\pi k)|^2 \sum_{k=-\infty}^{\infty}|W(\omega+2\pi k)|^2}, & \omega \in \mathcal{I}; \\ 0 & \omega \notin \mathcal{I}. \end{cases} \quad (8)$$

Here $\mathcal{I}$ is the set of frequencies ω for which $\sum_k |W(\omega - 2\pi k)|^2 \neq 0$ and $\sum_k |S(\omega - 2\pi k)|^2 \neq 0$, and $W(\omega)$ and $S(\omega)$ denote the continuous-time Fourier transforms of $w(t)$ and $s(t)$, respectively. The sampling scheme of Fig. 1 then reduces to the scheme depicted in Fig. 2.

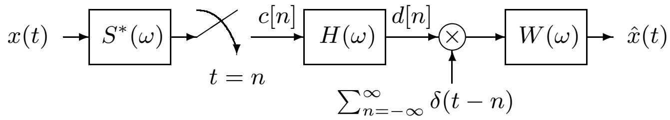

Figure 2: Minimax regret reconstruction of $x(t)$ in SI spaces.

3. ERROR ANALYSIS

We now analyze the error $e_{\text{REG}} = x - P_{\mathcal{W}} P_{\mathcal{S}} x$ of the minimax regret reconstruction, and compare it with the error of the consistent reconstruction $e_{\text{CON}} = x - E_{\mathcal{W}\mathcal{S}^\perp} x$. In particular, we develop bounds on the norm of the reconstruction error in both cases, and show, that if the spaces $\mathcal{S}$ and $\mathcal{W}$ are far enough apart, or if x has enough energy in $\mathcal{S}$, then the worst-case and best-case error using the minimax regret reconstruction are smaller than the worst-case and best-case error using the consistent reconstruction.

Before proceeding to the detailed development, we note that if $x \in \mathcal{W}$, then the consistent reconstruction will yield perfect reconstruction of x, which is not true for the minimax regret reconstruction. Therefore, if we know that $x \in \mathcal{W}$, then we should use the consistent reconstruction algorithm. The interesting question is how to reconstruct the signal when x does not necessarily lie in $\mathcal{W}$. In this case we now show that we can often reduce the error by using the minimax regret reconstruction. These analytical results are also demonstrated through simulation in Section 4.

Since the samples $c = S^* x$ are obtained with sampling vectors that are in $\mathcal{S}$, $c = 0$ for any $x \in \mathcal{S}^\perp$, and therefore the reconstruction error for any such x is equal to the norm of x, regardless of the choice of H. Thus, instead of developing bounds on the norm of the reconstruction error, we consider the ratios $\gamma_{\text{REG}} = \|e_{\text{REG}}\|^2 / \|P_{\mathcal{S}}^\perp x\|^2$ and $\gamma_{\text{CON}} = \|e_{\text{CON}}\|^2 / \|P_{\mathcal{S}}^\perp x\|^2$ where we assume explicitly throughout this section that $x \notin \mathcal{S}^\perp$.

Denoting by $e_{\text{OPT}} = P_{\mathcal{W}^\perp} x$ the error attainable when the sampling method is not restricted *i.e.,* $\mathcal{S} = \mathcal{W}$, we have that

$$\|e_{\text{REG}}\|^2 = \|e_{\text{OPT}}\|^2 + \|P_{\mathcal{W}} P_{\mathcal{S}^\perp} x\|^2 . \tag{9}$$

Note that for $x \in \mathcal{S}$, $\|e_{\text{REG}}\|^2 = \|e_{\text{OPT}}\|^2$, so that the minimax regret reconstruction is optimal. If $x \notin \mathcal{S}$, then

$$\gamma_{\text{REG}} = \gamma_{\text{OPT}} + \frac{\|P_{\mathcal{W}} P_{\mathcal{S}^\perp} x\|^2}{\|P_{\mathcal{S}^\perp} x\|^2}, \tag{10}$$

where $\gamma_{\text{OPT}} = \|e_{\text{OPT}}\|^2 / \|P_{\mathcal{S}}^\perp x\|^2$. Using the fact that $P_{\mathcal{S}^\perp} x / \|P_{\mathcal{S}^\perp} x\| \in C_{\mathcal{S}^\perp}$, where for an arbitrary subspace $\mathcal{A}$, $C_{\mathcal{A}} \triangleq \{a;\ a \in \mathcal{A}, \|a\| = 1\}$, we have

$$\inf_{v \in C_{\mathcal{S}^\perp}} \|P_{\mathcal{W}} v\|^2 \leq \frac{\|P_{\mathcal{W}} P_{\mathcal{S}^\perp} x\|^2}{\|P_{\mathcal{S}^\perp} x\|^2} \leq \sup_{v \in C_{\mathcal{S}^\perp}} \|P_{\mathcal{W}} v\|^2 . \tag{11}$$

Now, from the definition of the angle between two closed subspaces of a Hilbert space[2] [1, 6],

$$\inf_{v \in C_{\mathcal{S}^\perp}} \|P_{\mathcal{W}} v\|^2 = \cos^2\left(\mathcal{S}^\perp, \mathcal{W}\right)$$
$$\sup_{v \in C_{\mathcal{S}^\perp}} \|P_{\mathcal{W}} v\|^2 \triangleq M^2\left(\mathcal{S}^\perp, \mathcal{W}\right) = 1 - \cos^2\left(\mathcal{W}, \mathcal{S}\right). \tag{12}$$

[2]Given two closed subspace $\mathcal{A}_1$ and $\mathcal{A}_2$ of a Hilbert space $\mathcal{H}$, $\cos\left(\mathcal{A}_1, \mathcal{A}_2\right) \triangleq \inf_{a \in C_{\mathcal{A}_1}} \|P_{\mathcal{A}_2} a\|$. Similarly, $M\left(\mathcal{A}_1, \mathcal{A}_2\right) \triangleq \sup_{a \in C_{\mathcal{A}_1}} \|P_{\mathcal{A}_2} a\| = \sqrt{1 - \cos\left(\mathcal{A}_1, \mathcal{A}_2^\perp\right)} = \sqrt{1 - \cos\left(\mathcal{A}_2, \mathcal{A}_1^\perp\right)}$.

Combining (10), (11) and (12),

$$\cos^2\left(\mathcal{S}^\perp, \mathcal{W}\right) \leq \gamma_{\text{REG}} - \gamma_{\text{OPT}} \leq M^2\left(\mathcal{S}^\perp, \mathcal{W}\right). \tag{13}$$

If $v \in \mathcal{S}^\perp$ achieves the maximum (minimum) angle with $\mathcal{W}$,[3] then with $x = v + s$ with any $s \in \mathcal{S}$, x achieves the upper (lower) bound of (13). Therefore, the bounds of (13) are tight.

An upper bound on the error using the consistent reconstruction method was developed in [1] using the fact that

$$e_{\text{OPT}} = P_{\mathcal{W}^\perp} e_{\text{CON}}. \tag{14}$$

From (14) we can also develop a lower bound which results in

$$\frac{1}{1 - \cos^2\left(\mathcal{S}^\perp, \mathcal{W}\right)} \leq \frac{\gamma_{\text{CON}}}{\gamma_{\text{OPT}}} \leq \frac{1}{\cos^2\left(\mathcal{W}, \mathcal{S}\right)}. \tag{15}$$

As in the case of the bounds (13), it can be shown that the bounds of (15) are tight.

3.1. Bound Comparison

We now develop conditions under which the worst-case (best-case) error using the consistent reinstruction is larger than the worst-case (best-case) error using the minimax regret reconstruction.

To ensure that the worst-case and best-case values of γ_{CON} are larger (respectively) than the worst-case and best-case values of γ_{REG}, we must have that $\frac{\gamma_{\text{OPT}}}{\cos^2(\mathcal{W}, \mathcal{S})} \geq \gamma_{\text{OPT}} + 1 - \cos^2\left(\mathcal{W}, \mathcal{S}\right)$, and $\frac{\gamma_{\text{OPT}}}{1 - \cos^2\left(\mathcal{S}^\perp, \mathcal{W}\right)} \geq \gamma_{\text{OPT}} + \cos^2\left(\mathcal{S}^\perp, \mathcal{W}\right)$, or, equivalently,

$$\gamma_{\text{OPT}} = \frac{\|P_{\mathcal{W}^\perp} x\|^2}{\|P_{\mathcal{S}^\perp} x\|^2} \geq \cos^2\left(\mathcal{W}, \mathcal{S}\right), \tag{16}$$

for the worst case and

$$\gamma_{\text{OPT}} = \frac{\|P_{\mathcal{W}^\perp} x\|^2}{\|P_{\mathcal{S}^\perp} x\|^2} \geq M^2\left(\mathcal{S}^\perp, \mathcal{W}^\perp\right), \tag{17}$$

for the best case. These results are intuitive: If most of the energy of x lies in $\mathcal{S}$ then γ_{OPT} tends to be very large and both inequalities hold. Similarly, if $\mathcal{W}$ and $\mathcal{S}$ are "far apart", then the bounds will be small, and again the inequalities will typically hold.

Since $\cos^2\left(\mathcal{W}, \mathcal{S}\right), M^2\left(\mathcal{S}^\perp, \mathcal{W}^\perp\right) \leq 1$ for all $\mathcal{S}$ and $\mathcal{W}$, it follows that if $\|P_{\mathcal{W}^\perp} x\| \geq \|P_{\mathcal{S}^\perp} x\|$ then both in the best case and in the worst case, the minimax regret reconstruction is better than the consistent reconstruction.

4. EXAMPLE

We now consider an example illustrating the performance of the minimax regret and consistent reconstruction methods.

We consider the case in which we sample a continuous-time signal $x(t)$ using a non-ideal sampler, so that the samples $c[n]$ are equal to the average of the signal over intervals of length Δ:

$$c[n] = \frac{1}{\Delta} \int_{nT-\Delta}^{nT} x(t) dt. \tag{18}$$

[3]The inf and the sup in the definition of the angle can be replaced by min and max respectively, assuming $C_{\mathcal{S}^\perp}$ is a compact set.

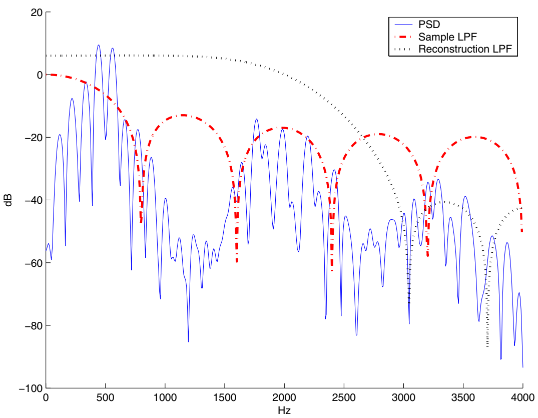

Figure 3: PSD of the speech fragment and the frequency responses of the non ideal low-pass filters.

The samples $c[n]$ can be obtained by filtering the signal $x(t)$ with a filter whose impulse response $s(t)$ is given by

$$s(t) = \begin{cases} \frac{1}{\Delta}, & 0 \leq t \leq \Delta; \\ 0, & \text{otherwise}, \end{cases} \tag{19}$$

and then sampling the output at times $t = nT$. The filter $s(t)$ can be viewed as a (non-ideal) low-pass filter (LPF). In the simulations below, we use $T = 4000^{-1}$[sec] and $\Delta = 1.125$[ms]. The reconstructed output $\hat{x}(t)$ is obtained from the transformed sequence $d = Hc$ using an interpolation kernel $w(t)$, *i.e.*, $\hat{x}(t) = \sum_n d[n]w(t - nT)$. Specifically, we consider $w(t)$ to be a non-ideal LPF with support on $t \in [0, 1.75]$[ms] which approximates an ideal LPF with cutoff frequency of 2[kHz].

For the purpose of simulation we approximate the continuous-time signal $x(t)$ with a discrete sequence $x[n]$ on a fine grid. In particular, we have used a speech fragment, taken from the Timit database [7], at a sample rate of 8[kHz]. The continuous time integration kernel $s(t)$ is approximated by the discrete filter

$$s[n] = \begin{cases} \frac{1}{N}, & 0 \leq n \leq N - 1; \\ 0, & \text{otherwise}, \end{cases} \tag{20}$$

with $N = 10$ samples. The ideal sampling is implemented by down-sampling the filter output with a decimation factor of 2. The (non-ideal) LPF followed by decimation can be described by proper construction of the sampling matrix S^*.

To implement the reconstruction we use a linear-phase FIR filter of order 14 (with cutoff frequency 2[kHz]) as the interpolation kernel. Here as well, the discrete-time interpolation kernel simulates the continuous-time interpolation kernel, by constructing it over the 8[kHz] fine grid and up-sampling the input sequence d by a factor of 2, prior to filtering. The up-sampling followed by the filtering operation can be described by properly constructing the matrix W.

In Figure 3 we plot the power spectrum density estimate of the input speech signal and the frequency responses of the sampling and the reconstruction kernels. Figure 4 presents the input sequence $x[n]$ and 3 different reconstructed signals, corresponding to $H = I$ (that is, not applying a correcting transformation at all), $H = (S^*W)^\dagger$ (consistent reconstruction) and $H = H_{\text{REG}}$ of Theorem 1. As can be seen from the figure, the results of direct reconstruction are poor. The consistent reconstruction and minimax reconstruction methods perform much better. However, it can be seen that the minimax regret reconstruction leads to better results than the consistent reconstruction method. Computing the error $\|x - \hat{x}\|$ for this example, we obtain that the errors resulting form direct, consistent and minimax regret reconstruction are 2.5977, 1.9111 and 1.2101, respectively.

Similar results where obtained for other choices of the input signal. Thus, not only does the minimax regret estimator minimize the worst-case error, it seems to lead to superior results for other choices of x as well.

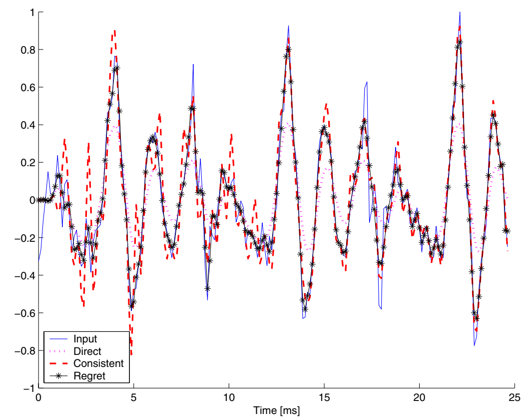

Figure 4: The original speech fragment $x[n]$ and the reconstructed signal using three different methods: direct reconstruction, consistent reconstruction and the minimax regret reconstruction.

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