Geometrical Interpretation of Consistent Sampling

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Abstract

We first introduce the concept of consistent sampling in infinite dimensional spaces and derive a general mathematical framework. The underlying linear reconstruction scheme coincides with the so-called oblique projection, which turns into an ordinary orthogonal projection when adapting the inner product. The inner product of interest is, in general, not unique. We characterize the inner products and corresponding positive operators for which the new geometrical interpretation applies. Finally, we study explicit constructions of such positive operators by means of Riesz bases.

1 Consistent Sampling

Suppose we are given measurements c_i of an unknown signal f belonging to a linear class $\mathcal{H}$ endowed with an inner product $\langle \cdot, \cdot \rangle$. The measurements are assumed to be of the form

$$c_i = \langle f, s_i \rangle$$

for some function $f \in \mathcal{H}$ and a set of sampling vector $\{s_i\}$ that span a subspace $\mathcal{S} \subseteq \mathcal{H}$. We want to reconstruct f from the observations c_i using a given set of reconstruction vectors $\{w_i\}$ that span a subspace $\mathcal{W} \subseteq \mathcal{H}$ [1, 4].

For designing the reconstruction algorithm we start with the following natural requirements on the sampling and reconstruction:

1. uniqueness of signal $\hat{f} \in \mathcal{W}$ with $\langle \hat{f}, s_i \rangle = c_i$,

2. consistent sampling (interpolation) [6] in the sense that $\langle \hat{f}, s_i \rangle = \langle f, s_i \rangle$.

The first requirement is a requirement on the sampling process. Specifically, we want the sampling vectors s_i to be such that if $\langle f_1, s_i \rangle = \langle f_2, s_i \rangle$ for all i, where $f_1, f_2 \in \mathcal{W}$, then $f_1 = f_2$ or, equivalently,

$$\forall f_1, f_2 \in \mathcal{W}, \quad \langle f_1 - f_2, s_i \rangle = 0 \Rightarrow f_1 = f_2. \tag{1}$$

To satisfy (1), we must have that

$$\mathcal{W} \cap \mathcal{S}^{\perp} = \{0\}\,. \tag{2}$$

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If the second requirement is also satisfied, so that $\langle \hat{f}, s_i \rangle = \langle f, s_i \rangle$, then for any $f \in \mathcal{W}$, we must have that $\hat{f} = f$. Otherwise we will have two signals f and $\hat{f}$ in $\mathcal{W}$, that have the same measurements, which contradicts the uniqueness requirement.

In our development, we focus our attention on linear reconstruction. In this case, the reconstruction $\hat{f}$ is of the form

$$\hat{f} = \sum_i d_i w_i$$

for some coefficients d_i that are a linear transformation of the measurements c_i. In other words,

$$d = Hc$$

for some linear mapping H or, with S and W denoting the synthesis operators corresponding to $\{s_i\}$ and $\{w_i\}$, respectively,

$$\hat{f} = Wd = WHc = WHS^* f\,.$$

For practical purposes, we restrict our attention to the case in which

$$\mathcal{H} = \mathcal{W} \oplus \mathcal{S}^\perp\,. \tag{3}$$

Since the spaces $\mathcal{S}$ and $\mathcal{W}$ satisfy (2), we can always define $\mathcal{H}$ as in (3).

It was shown in [3], that under the assumption (3), a linear reconstruction is consistent if and only if it has the form $\hat{f} = E_{\mathcal{W}\mathcal{S}^\perp} f$, where $E_{\mathcal{W}\mathcal{S}^\perp}$ is the oblique projection onto $\mathcal{W}$ along $\mathcal{S}$, and is the unique operator satisfying

$$\begin{aligned} E_{\mathcal{W}\mathcal{S}^\perp} w &= w, \qquad (w \in \mathcal{W}), \\ E_{\mathcal{W}\mathcal{S}^\perp} v &= 0, \qquad (v \in \mathcal{S}^\perp). \end{aligned}$$

Specifically, we have the following theorem.

Theorem 1.1. [3] *Let $\{c_i = \langle f, s_i \rangle\}$ denote measurements of $f \in \mathcal{H}$ with sampling vectors $\{s_i\}$ that span a subspace $\mathcal{S} \subseteq \mathcal{H}$, and let the reconstruction vectors $\{w_i\}$ span a subspace $\mathcal{W} \subseteq \mathcal{H}$ such that $\mathcal{H} = \mathcal{W} \oplus \mathcal{S}^\perp$. Then, $\hat{f}$ is a linear consistent reconstruction of f if and only if $\hat{f} = E_{\mathcal{W}\mathcal{S}^\perp} f$.*

Theorem 1.1 establishes existence of the unique consistent reconstruction. In practice, we need an explicit method. As shown in [3], if the sets $\{s_i\}$ and $\{w_i\}$ consist of finitely many vectors, then consistent reconstruction is always possible. In the case in which both sets of vectors are linearly independent,

$$E_{\mathcal{W}\mathcal{S}^\perp} = W(S^*W)^{-1}S^*\,,$$

and the reconstruction is given by

$$\hat{f} = W(S^*W)^{-1}c\,.$$

If the sets are linearly dependent, i.e., redundant, then the inverse operation is replaced by the pseudoinverse [3].

In an infinite-dimensional setting, stability of the reconstruction becomes an issue. In this case, as we now show, Riesz-bases and frames are the right concepts generalizing linearly independent and redundant systems in order to overcome instability problems and ensure the existence of a consistent reconstruction. Due to space limitation, here we focus only on the linearly independent case.

In order to derive a stable reconstruction scheme we need some more assumptions on the basic systems of $\mathcal{S}$ and $\mathcal{W}$.

Lemma 1.2. *Let $\mathcal{S}$ and $\mathcal{W}$ be closed subspaces of a Hilbert space $\mathcal{H}$ with Riesz bases $\{s_i\}_{i\in I}$ and $\{w_i\}_{i\in I}$ defining the synthesis operators S and W, respectively. The following statements are equivalent. Then, $\mathcal{H} = \mathcal{W} \oplus \mathcal{S}^\perp$ if and only if S^*W is invertible on $l^2(K)$ to $l^2(J)$.*

With the assumptions of Lemma 1.2 and $\mathcal{H} = \mathcal{W} \oplus \mathcal{S}^\perp$, it follows from the inverse mapping theorem [2] that $(S^*W)^{-1}$ is bounded, and we have that

$$\hat{f} = W(S^*W)^{-1}c$$

is a stable consistent reconstruction of f, as in the finite-dimensional case. Similar results hold when replacing Riesz bases by frames.

2 Geometrical Interpretation

We now develop a geometrical interpretation of the consistent reconstruction.

In the special case in which $\mathcal{W} = \mathcal{S}$, $\hat{f} = P_{\mathcal{W}} f$ is equal to the orthogonal projection of f onto $\mathcal{W}$, or, equivalently, it is the signal in $\mathcal{W}$ closest to f. It is therefore not surprising that if we change the inner product of $\mathcal{H}$ such that the oblique projection turns into an orthogonal projection, then the resulting reconstruction is again the minimal norm signal in $\mathcal{W}$, for the changed norm. We therefore seek an inner product which turns $\mathcal{W}$ and $\mathcal{S}^\perp$ into orthogonal spaces.

Definition 2.1. *Let $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ be two inner products for a Hilbert space $\mathcal{H}$. They are said to be equivalent if there exist $0 < a \le b$ such that*

$$a\langle h, h\rangle_1 \le \langle h, h\rangle_2 \le b\langle h, h\rangle_1, \qquad (h \in \mathcal{H}).$$

In what follows A always denotes a bounded linear positive operator onto $\mathcal{H}$. Such an operator defines an equivalent inner product on $\mathcal{H}$ by

$$\langle h, k\rangle_A = \langle h, Ak\rangle, \qquad (h, k \in \mathcal{H}).$$

Conversely, for every equivalent inner product $\langle \cdot, \cdot \rangle_2$ there exists a unique operator A such that

$$\langle h, k\rangle_2 = \langle h, Ak\rangle, \qquad (h, k \in \mathcal{H})\,.$$

Our problem therefore is to find a continuous positive operator A such that the resulting inner product turns $\mathcal{W}$ and $\mathcal{S}^\perp$ into orthogonal spaces. We arrive at the following result.

Theorem 2.2. *Let $\mathcal{W}$ and $\mathcal{S}^\perp$ denote subspaces of a Hilbert space $\mathcal{H}$ such that $\mathcal{H} = \mathcal{W} \oplus \mathcal{S}^\perp$, and let A denote an arbitrary positive operator. Then the orthogonal projection onto $\mathcal{W}$ with respect to the inner product $\langle \cdot, \cdot \rangle_A$ satisfies $P^A_{\mathcal{W}} = E_{\mathcal{W}\mathcal{S}^\perp}$ if and only if $A : \mathcal{W} \to \mathcal{S}$, or equivalently, $A : \mathcal{S}^\perp \to \mathcal{W}^\perp$.*

Theorem 2.2 characterizes the class of operators that turn $\mathcal{W}$ and $\mathcal{S}^\perp$ into orthogonal spaces. As can easily be seen, this class is closed under linear combinations with positive coefficients.

3 From Oblique to Orthogonal Projection

If $P^A_{\mathcal{W}}$ denotes the orthogonal projection onto $\mathcal{W}$ with respect to the inner product defined by any operator A with $A : \mathcal{W} \to \mathcal{S}$, then we have

$$E_{\mathcal{W}\mathcal{S}^\perp} = P^A_{\mathcal{W}}.$$

What we see is that the oblique projection performs an orthogonal projection with respect to a whole class of inner products. In other words, the consistent solution is the minimal norm interpolation of the data c_i for the corresponding sampling vectors s_i, with respect to any norm defined by such a positive operator A. The interesting observation is that although we have several different but equivalent norms, the minimal norm interpolation always remains the same.

Assume that we have a Riesz basis $\{w_i\}$ of $\mathcal{W}$ whose synthesis operator will be denoted by W. Our next goal is to find an inner product on $\mathcal{H}$ which turns $\{w_i\}$ into an orthonormal basis of $\mathcal{W}$, out of the class of all inner products which make $\mathcal{W}$ orthogonal to $\mathcal{S}^\perp$. If A defines such an inner product, then we have

$$E_{\mathcal{W}\mathcal{S}^\perp} = P^A_{\mathcal{W}} = WW^{*_A}, \tag{4}$$

where W^{*_A} is the adjoint with respect to the inner product defined by A, and is given by $W^{*_A} = W^* A$.

For Riesz bases $\{s_i\}$, $\{w_i\}$, $\{s_i^\perp\}$, and $\{w_i^\perp\}$ for the spaces $\mathcal{S}$, $\mathcal{W}$, $\mathcal{S}^\perp$, and $\mathcal{W}^\perp$, respectively, with corresponding synthesis operators S, W, $S^\perp$, and $W^\perp$, the operator

$$A = S(W^*S)^{-1}(S^*W)^{-1}S^* + W^\perp(S^{\perp *}W^\perp)^{-1}(W^{\perp *}S^\perp)^{-1}W^{\perp *}$$

defines an inner product on $\mathcal{H}$ for which $\{w_i\}$ turns into an orthonormal basis and $\mathcal{W}$ is orthogonal to $\mathcal{S}^\perp$.

There are many other operators A satisfying (4). However, it is interesting to observe that the action of such operators on $\mathcal{W}$ is unique. Indeed, since for any $w \in \mathcal{W}$, $w = WW^{*_A}w = WW^{*_B}w$, we have that

$$Aw = Bw, \qquad (w \in \mathcal{W}).$$

Thus any operator B satisfying $E_{\mathcal{W}\mathcal{S}^\perp} = WW^{*_B}$ must have the form

$$B = S(W^*S)^{-1}(S^*W)^{-1}S^* + C,$$

where $Cw = 0$ for any $w \in \mathcal{W}$. In addition, to satisfy Theorem 2.2, C must map $\mathcal{S}^\perp$ to $\mathcal{W}^\perp$. We therefore have the following theorem.

Theorem 3.1. *Let $\mathcal{W}$ and $\mathcal{S}^\perp$ denote subspaces of a Hilbert space $\mathcal{H}$ such that $\mathcal{H} = \mathcal{W} \oplus \mathcal{S}^\perp$, let $\{w_i\}$ denote a Riesz basis for $\mathcal{W}$ corresponding to W, and let A denote an arbitrary positive operator. Then $E_{\mathcal{W}\mathcal{S}^\perp} = P^A_{\mathcal{W}} = WW^{*_A}$ if and only if $A = S(W^*S)^{-1}(S^*W)^{-1}S^* + C$ where $C : \mathcal{W} \to \{0\}$ and $C : \mathcal{S}^\perp \to \mathcal{W}^\perp$.*

It is interesting to note that any operator A satisfying the requirements of Theorem 3.1 has the additional property that W^{*_A} is the oblique pseudoinverse on l_2 along $\mathcal{S}^\perp$ [5].

Let $\{s_i\}$ be a Riesz basis for $\mathcal{S}$. If we want to linearly reconstruct a signal $f \in \mathcal{H}$ in $\mathcal{W}$ from the samples $\langle f, s_i\rangle$ requiring consistency, then we recall that

$$\hat{f} = E_{\mathcal{W}\mathcal{S}^\perp} f = W(S^*W)^{-1}(S^*f),$$

which is equivalent to the orthogonal projection

$$\hat{f} = P^A_{\mathcal{W}} f = \sum_i \langle f, Aw_i\rangle w_i \tag{5}$$

for an operator A satisfying the properties of Theorem 3.1. Here, we see that for reconstruction, A needs only to be applied on $\mathcal{W}$, so that for all practical purposes A is uniquely defined. It follows from (5) that $\hat{f}$ is a minimal norm approximation to f with respect to the inner product defined by A. Specifically, for any $w \in \mathcal{W}$, $\|f - w\|_A \geq \|f - \hat{f}\|_A$.

References

[1] A. Aldroubi and M. Unser, Sampling procedures in function spaces and asymptotic equivalence with Shannon's sampling theory. *Numer. Funct. Anal. Optimiz.*, 15:1–21, Feb. 1994.

[2] J. B. Conway, A Course in Functional Analysis. *Springer-Verlag.*, 1985.

[3] Y. C. Eldar, Sampling and reconstruction in arbitrary spaces and oblique dual frame vectors. *J. Fourier Analys. Appl.*, 1(9):77–96, Jan. 2003.

[4] S. J. Mallat, A theory of multiresolution signal decomposition. *IEEE Trans. Pattern Anal. Mach. Intell.*, 11:674–693, 1989.

[5] R. D. Milne. An oblique matrix pseudoinverse. *SIAM J. Appl. Math.*, 16(5):931–944, Sep. 1968.

[6] M. Unser and A. Aldroubi A general sampling theory for nonideal acquisition devices. *IEEE Trans. Signal Processing*, 42(11):2915–2925, Nov. 1994.