Detection of an Ensemble Mixed with Unknown States
Noam Elron and Yonina C. Eldar
Department of Electrical Engineering, Technion - Israel Institute of Technology

What Is a Quantum Detector?
A quantum detector is a measurement designed to discriminate between a finite set of quantum states.

INPUT: \( m \) states \( \rho_i \) with prior probabilities \( p_i \)
\[ \rho = \sum_{i=1}^{m} p_i \rho_i \]

A detector is a POVM \( \Pi = \{ \Pi_i \}_{i=1}^{m} \) defined so that
\[ P_{\Pi_i} \{ \text{Identify} \, \rho_i \} = \text{Tr}(\Pi_i \rho_i) \]

The Probability of Correct Detection
\[ P_D = \sum_{i=1}^{m} p_i \text{Tr}(\Pi_i \rho_i) \]

An Optimal Detector - Maximal \( P_D \)
Solve the optimization problem
\[ \max_{\Pi \text{ is a POVM}} P_D(\Pi) = \max_{\Pi \text{ is a POVM}} \left\{ \sum_{i=1}^{m} p_i \text{Tr}(\Pi_i \rho_i) \right\} \]
subject to: \( \Pi_i \geq 0 \), \( \sum_{i=1}^{m} \Pi_i = I \)

• Necessary and sufficient conditions for optimality are known (but not very practical).
• This problem is solvable using iterative algorithms with polynomial complexity.

Uncertain States
One never has full knowledge of physical parameters.

We examine states which are a mixture of a known state \( \rho_i \) and an unknown state \( \rho'_i \):
\[ \rho_i = q_i \rho_i + (1-q_i) \rho'_i \]

• The mixing bounds \( q_i \) are known.
• All we assume about \( \rho'_i \) is that they are valid quantum states.

The Nominal Ensemble
The known states \( \rho_i \) with prior probabilities \( p_i \).

Detection with Uncertainty
How does one go about designing a “good” detector without fully knowing the states \( \rho_i \)?

We must redefine our criterion of optimality.

New Criterion
\[ \max_{\Pi \text{ is a POVM}} \left\{ \min_{\rho_i \neq \rho_j} P_D(\Pi, \rho_i) \right\} \]

Optimal Worst-Case Detection
\[ \min_{\rho_i \neq \rho_j} P_D(\Pi, \rho_i) = \text{Tr}(\Pi_i \rho_i) \]

Equivalent to Maximization of
\[ P_D^{wc}(\Pi) = \sum_{i=1}^{m} p_i \text{Tr}(\Pi_i |q_i \rho_i + (1-q_i) \rho'_i|) \]

The Optimal Detector
* The Lower Bound Detector
* The optimal detector for the effective states \( \rho_i = q_i \rho_i + (1-q_i) \rho'_i \) with prior probabilities \( p_i \).

Uniform Uncertainty
\[ (q_i = q) \]

The Lower Bound Detector
Arbitrarily choose the state with maximal prior probability.
Achieves \( P_D = \rho_{\text{nom}} \)
It does not depend on \( \rho_i \)
A lower bound on all “optimal” detectors.

Performance Analysis - Uniform Uncertainty

Fig. 3: Performance of the different optimal detectors under uniform uncertainty, as a function of \( q \).

Fig. 4: Measurement distances (top) and their derivatives under uniform uncertainty, as a function of \( q \).

Analysis of Figures 3 & 4
• Simulation: 3 non-orthogonal pure states, with \( p_{\text{nom}} = 0.1 \). Uniform Uncertainty.
• Optimality under our criterion does not grant good performance throughout the region of uncertainty.
• For low \( q \), both detectors coincide with the Lower Bound Detector.
• The transition to \( \Pi^{wc} \) is instantaneous.
• \( \Pi^{wc} \) exhibits both continuous and discontinuous change.
• For high \( q \), \( \Pi^{wc} = \Pi^{nom} \). This is a common but not a universal trait.

Conclusions
• Uncertainty reduces the probability of correct detection.
• The lower bound appears explicitly in the Optimal WC Detector, and implicitly in the Optimal Average Detector.
• The uncertainty is manifested as an “effective ensemble”.
• The characteristics of the effective ensemble are determined by the criterion of optimality.
• The optimal detector is biased towards the more certain states.
• Under symmetrical conditions the bias cancels out and one can simply ignore the uncertainty.