Detection of an Ensemble Mixed with Unknown States

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What Is a Quantum Detector?

A quantum detector is a measurement designed to discriminate between a finite set of quantum states.

INPUT: m states ρ_i with prior probabilities p_i

$$\rho = \sum_{i=1}^{m} p_i \rho_i$$

A detector is a POVM $\Pi = \{\Pi_i\}_{i=1}^m$ defined so that

$$\Pr\left\{ \underset{\rho_i}{\text{Identify}} \;\middle|\; \rho = \rho_i \right\} = \mathrm{Tr}(\Pi_i \rho_i)$$

The Probability of Correct Detection

$$P_d \triangleq \sum_{i=1}^{m} p_i \mathrm{Tr}(\Pi_i \rho_i)$$

An Optimal Detector - Maximal P_d

Solve the optimization problem

$$\max_{\Pi \text{ is a POVM}} P_d(\Pi) \quad \Rightarrow \quad \max_{\Pi_i} \left\{ \sum_{i=1}^{m} p_i \mathrm{Tr}(\Pi_i \rho_i) \right\}$$

$$\text{s.t.:} \quad \Pi_i \geq 0; \quad \sum_{i=1}^{m} \Pi_i = I$$

- Necessary and sufficient conditions for optimality are known (but not very practical).
- This problem is solvable using iterative algorithms with polynomial complexity.

Uncertain States

One never has full knowledge of physical parameters.

We examine states which are a mixture of a known state ρ_i^0 and an unknown state ρ_i^1.

$$\rho_i = q_i \rho_i^0 + (1 - q_i)\rho_i^1$$

- The mixing bounds q_i are known.
- All we assume about ρ_i^1 is that they are valid quantum states.

The Nominal Ensemble

The known states ρ_i^0 with prior probabilities p_i.

Detection with Uncertainty

How does one go about designing a "good" detector without fully knowing the states ρ_i?

We must redefine our criterion of optimality.

New Criterion

$$\max_{\Pi \text{ is a POVM}} \left\{ \min_{\rho_i^1 \text{ are QS}} P_d(\Pi, \rho_i^1) \right\}$$

Equivalent to Maximization of

$$P_d^{wc}(\Pi) = \sum_{i=1}^{m} p_i \mathrm{Tr}[\Pi_i (q_i \rho_i^0 + (1 - q_i)\lambda_{\Pi_i}^{\min} I)]$$

The Optimal Detector

* The Lower Bound Detector - we cannot do better than guess

or

* The optimal detector for the known states ρ_i^0 with *effective prior probabilities* $\tilde{p}_i \triangleq p_i q_i / \sum_i p_i q_i$

Uniform Uncertainty ($q_i = q$)

high uncertainty: $\hat{\Pi}^{wc}$ = Lower Bound Detector

low uncertainty: $\hat{\Pi}^{wc} = \hat{\Pi}^{nom}$

Optimal *Worst-Case* Detection

Optimal *Average* Detection

Assume ρ_i^1 are:
* random quantum states
* statistically independent
* uniformly distributed.

Find the measurement that achieves

$$\max_{\Pi \text{ is a POVM}} \left\{ E_{\rho_i^1} P_d(\Pi, \rho_i^1) \right\}$$

$$P_d^{avg}(\Pi) = \sum_{i=1}^{m} p_i \mathrm{Tr}[\Pi_i (q_i \rho_i^0 + \tfrac{1-q_i}{n} I)]$$

The optimal detector for the *effective states* $\rho_i = q_i \rho_i^0 + \frac{1-q_i}{n} I$ with prior probabilities p_i

If $\forall i \;\; p_i = \frac{1}{m}$ then

$$\hat{\Pi}^{avg} = \hat{\Pi}^{nom}$$

The Lower Bound Detector: Arbitrarily choose the state with maximal prior probability.

Achieves $P_d = p_{\max}$.

It does not depend on ρ_i $\Rightarrow$ A lower bound on all "optimal" detectors.

Performance Analysis - General

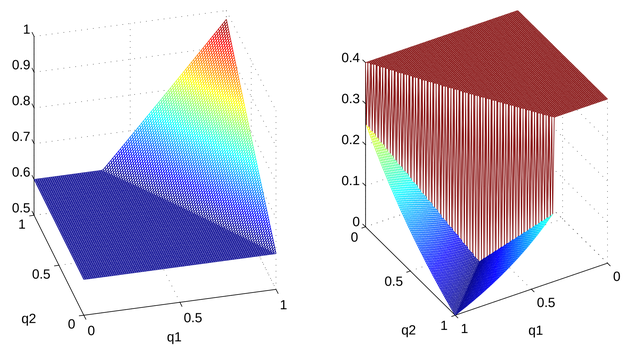

Fig. 1: $\hat{P}_d^{wc}$ (left) and $\mathrm{dist}(\hat{\Pi}^{wc}, \hat{\Pi}^{nom})$ (right) as a function of q_i

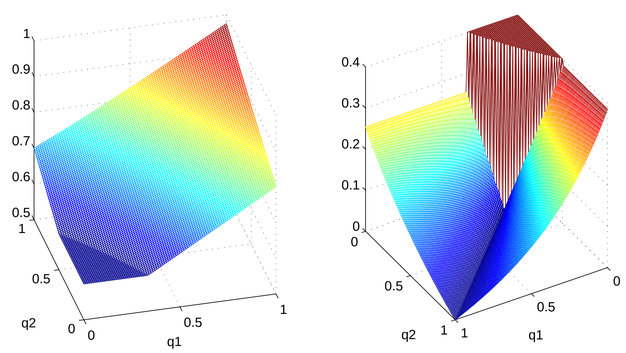

Fig. 2: $\hat{P}_d^{avg}$ (left) and $\mathrm{dist}(\hat{\Pi}^{avg}, \hat{\Pi}^{nom})$ (right) as a function of q_i

Analysis of Figures 1 & 2

- *Simulation: 2 non-orthogonal pure states, with $p_{\max} = p_2 = 0.6$.*
- *For clear presentation the q axes of the righthand graphs are rotated.*
- *Both probs. of corr. det. are constant at $p_{\max}$ under high uncertainty (low q_i).*
- *In this region, $D(\cdot, \hat{\Pi}^{nom})$ are constant, and exhibit discontinuity on the edges - transition between the Lower Bound Det. and a ρ_i^0 dependent detector.*
- *For $q_1 = q_2 > q^*$, $D(\cdot, \hat{\Pi}^{nom}) = 0$ - see 'uniform uncertainty'.*

Performance Analysis - Uniform Uncertainty

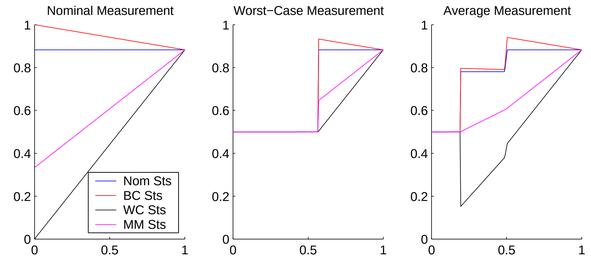

Fig. 3: Performance of the different optimal detectors under uniform uncertainty, as a function of q

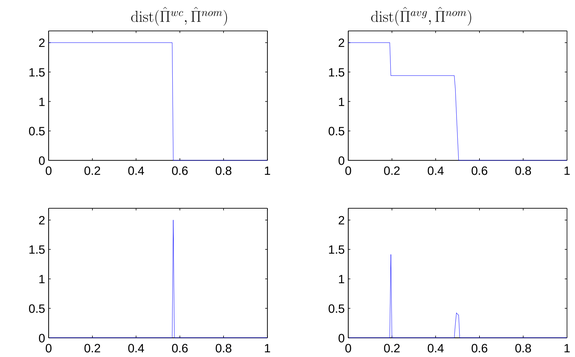

Fig. 4: Measurement distances (top) and their derivatives under uniform uncertainty, as a function of q

Analysis of Figures 3 & 4

- *Simulation: 3 non-orthogonal pure states, with $p_{\max} = 0.5$. Uniform Uncertainty.*
- *Optimality under one criterion does not grant good performance throughout the region of uncertainty.*
- *For low q, both detectors coincide with the Lower Bound Detector.*
- *The transition in $\hat{\Pi}^{wc}$ is instantaneous.*
- *$\hat{\Pi}^{avg}$ exhibits both continuous and discontinuous change.*
- *For high q, $\hat{\Pi}^{avg} = \hat{\Pi}^{nom}$. This is a common but not a universal trait.*

Conclusions

- Uncertainty reduces the probability of correct detection.
- The lower bound appears explicitly in the Optimal WC Detector, and implicitly in the Optimal Average Detector.
- The uncertainty is manifested as an "effective ensemble".
- The characteristics of the effective ensemble are determined by the criterion of optimality.
- The optimal detector is biased towards the more certain states.
- Under symmetrical conditions the bias cancels out and one can simply ignore the uncertainty.

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