# ROBUST PEAK DISTORTION EQUALIZATION

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## ABSTRACT

We consider the problem of designing a robust linear equalizer under channel uncertainties. Specifically, we propose a robust peak distortion (RPD) equalizer, in which we minimize the error probability of the worst-case sequence and the worst-case channel, in the uncertainty region. We show that the RPD equalizer can be found efficiently using standard convex optimization packages. We then demonstrate through simulations that under channel uncertainty the RPD equalizer outperforms traditional equalizers, and previously proposed robust equalizers.

## 1. INTRODUCTION

Channel equalization methods are used to combat the effects of inter-symbol interference (ISI) that results from the frequency and time selectivity characteristics of the transmission channel. The optimal equalization method, which minimizes the bit error rate (BER), has exponential computational complexity. Therefore, various sub-optimal approaches are used in practice, including linear equalization (LE) [1,2], and nonlinear decision feedback equalization (DFE) [3]. The most common methods for designing linear equalizers are based on the zero forcing (ZF) and the minimum mean squared error (MMSE) criteria, which lead to a closed form solution. However, the ZF and MMSE criteria do not aim to minimize the error probability directly. An alternative design method is the peak distortion (PD) criterion [1,4], which aims to minimize the error probability of the worst-case transmitted sequence. As is shown in [4], in the case that the error probability is sufficiently low, the PD equalizer minimizes the error probability.

Each of the methods described above assumes that the transmission channel is perfectly known. However, in most practical cases, the transmission channel is known with limited accuracy, for example, in the case that the channel is estimated with errors, in the case that the channel is time varying or in the case that the channel coefficients have some precision limitation. In these cases the standard LE and DFE equalizers employing the PD, MMSE and ZF criteria may not achieve the designated optimality.

To account for the channel uncertainty, it is desirable to design equalizers that have good performance across all possible channels in the region of uncertainty. In [5], robust LE and DFE were proposed, based on the MMSE and ZF criteria. Note, however, that these methods do not optimize the error probability. In this paper we propose a new robust LE, which is based on the PD criterion. For sufficiently low error probability, this equalizer minimizes the worst-case error probability across all possible channels in the region of uncertainty and all possible symbol sequences. The proposed equalizer, which is based on the potent framework of convex optimization theory [6,7], is shown through simulations to outperform traditional equalizers, and previously proposed robust equalizers.

The paper is organized as follows. In Section 2, we formulate our problem and review the basics of linear equalization. The peak distortion equalizer is reviewed in Section 3. In Section 4 we develop the robust peak distortion equalizer. In Section 5 we demonstrate its advantages over traditional methods.

## 2. LINEAR EQUALIZATION

Consider a communication system with additive white Gaussian noise (AWGN). The discrete-time model for the received, equivalent low-pass [2], signal is given by:

$$z_n = \sum_{i=0}^{K} h_i a_{n-i} + w_n , \quad (1)$$

where $a_i$ is the transmitted symbol at time index i, $h_i$ are the impulse response coefficients of the propagation channel, $w_n$ is a zero-mean AWGN process with variance $\sigma^2$, and $z_n$ is the channel's output at time index *n*. For simplicity we assume that all symbols are equally likely and that $a_i \in \{1,-1\}$. Furthermore, we assume that $h_i$ and $w_n$ are real-valued. The extension to higher-dimensional signal constellations and complex valued noise and channel coefficients is straightforward.

The LE aims to compensate for the channel and noise using a linear finite impulse response (FIR) filter. The output of the length-(2L+1) equalizer is given by:

$$\begin{aligned} y_n &= \sum_{j=-L}^{L} g_j z_{n-j} \\ &= \sum_{j=-L}^{L} \sum_{i=0}^{K} g_j h_i a_{n-j-i} + \sum_{j=-L}^{L} g_j w_{n-j} , \end{aligned} \quad (2)$$

where $\{g_j\}_{j=-L}^{L}$ are the equalizer filter taps. For convenience, we rewrite (2) in vector form:

$$y_n = \mathbf{a}^T \mathbf{H} \mathbf{g} + \mathbf{g}^T \mathbf{w} \quad (3)$$

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where

$$\mathbf{a} = [a_{n-L-K} \quad \cdots \quad a_{n+L}]^T, \qquad \mathbf{g} = [g_{-L} \quad \cdots \quad g_L]^T, \tag{4}$$

and **H** is a $(2L+K+1)\times(2L+1)$ Toeplitz convolution matrix:

$$\mathrm{H} = \begin{bmatrix} h_0 & & \\ \vdots & \ddots & \\ h_K & \ddots & h_0 \\ & \ddots & \vdots \\ & & h_K \end{bmatrix}. \tag{5}$$

The n-th symbol is detected as $\hat{a}_n = \mathrm{sign}(\hat{a}_n)$.

For completeness we provide the MMSE and ZF equalizers. The MMSE equalizer is defined as:

$$\mathbf{g}_{MMSE} = \arg\min_{\mathbf{g}} \mathrm{E}_{a,w}\left\{(y_n - a_n)^2\right\}, \tag{6}$$

and is given by:

$$\mathbf{g}_{MMSE} = \left(\mathrm{H}^T\mathrm{H} + \sigma^2\mathrm{I}\right)^{-1}\mathrm{H}^T\mathbf{e}_D. \tag{7}$$

where $\mathbf{e}_D$ denotes a column zeros vector with a one in the D-th location. The ZF equalizer is defined as:

$$\mathbf{g}_{ZF} = \arg\min_{\mathbf{g}}\left\{\left\|\mathrm{H}\mathbf{g} - \mathbf{e}_D\right\|^2\right\}, \tag{8}$$

where $\|\cdot\|^2$ denotes the standard Euclidean norm. The solution to the ZF optimization (8), is given by:

$$\mathbf{g}_{ZF} = \left(\mathrm{H}^T\mathrm{H}\right)^{-1}\mathrm{H}^T\mathbf{e}_D. \tag{9}$$

Assuming that the ISI power is less than the power of the current symbol, i.e., the eye diagram is open:

$$\mathbf{a}^T\mathrm{H}\mathbf{g} \ge 0 \;\; \forall \mathbf{a} : a_D = 1, \tag{10}$$

where D accounts for the delay of the cascade of the channel **h** and the equalizer $\mathbf{g}$, the BER of a LE is given by [2]:

$$\overline{P}_\varepsilon(\mathbf{g}) = \mathrm{E}_a\left\{P_\varepsilon(\mathbf{g},\mathbf{a})\right\}, \tag{11}$$

where

$$P_\varepsilon(g,a) = Q\left(\sqrt{\frac{\left(\mathbf{a}^T\mathrm{H}\mathbf{g}\right)^2}{\sigma^2\mathbf{g}^T\mathbf{g}}}\right), \tag{12}$$

and the Q-function is defined as:

$$Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{t^2}{2}}dt. \tag{13}$$

Unfortunately, the function $\overline{P}_\varepsilon(\mathbf{g})$ is highly nonlinear and difficult to optimize. Hence, sub-optimal criteria are used instead. The most commonly used sub-optimal criteria are the ZF and MMSE criteria. Those criteria, however, do not attempt to minimize the error probability directly. A criterion, which minimizes the worst-case sequence error probability, is the PD, which we now discuss.

## 3. PEAK DISTORION EQUALIZATION

In this section we formulate the standard PD equalizer in a way that will provide further insight into our robust equalization method. Under the assumption that the error probability is sufficiently low, the sum of exponents, in the expectation for the error probability, can be approximated by the largest exponent so that (11) becomes:

$$\overline{P}_\varepsilon(\mathbf{g}) \approx \max_a Q\left(\sqrt{\frac{\left(\mathbf{a}^T\mathrm{H}\mathbf{g}\right)^2}{\sigma^2\mathbf{g}^T\mathbf{g}}}\right), \tag{14}$$

where the maximum is over all possible symbols sequences. When the error probability is sufficiently low, the optimum equalizer will minimize the error probability (14). Under the assumption that the eye diagram is open, the equalizer that minimize (14) can be formulated as:

$$\mathbf{g}_{PD} = \arg\begin{cases} \min_g \max_a Q\left(\sqrt{\dfrac{\left(\mathbf{a}^T\mathrm{H}\mathbf{g}\right)^2}{\sigma^2\mathbf{g}^T\mathbf{g}}}\right) \\ s.t. \quad \mathbf{a}^T\mathrm{H}\mathbf{g} \ge 0 \;\; \forall \mathbf{a} : a_D = 1 \end{cases}, \tag{15}$$

Each Q-function in (15) is monotonically decreasing with respect to the SINR, thus maximizing the Q-function in (15), is equivalent to minimizing the SINR, hence:

$$\mathbf{g}_{PD} = \arg\begin{cases} \max_g \min_a \dfrac{\left(\mathbf{a}^T\mathrm{H}\mathbf{g}\right)^2}{\sigma^2\mathbf{g}^T\mathbf{g}} \\ s.t. \quad \mathbf{a}^T\mathrm{H}\mathbf{g} \ge 0 \;\; \forall \mathbf{a} : a_D = 1 \end{cases}. \tag{16}$$

The $\mathbf{g}_{PD}$ (16) will minimize the worst-case SINR for every possible transmitted sequence. This is the precise definition of the PD equalizer. The optimal solution to (16) is unique up to a positive constant factor. Therefore, we can set:

$$\min_a\left(\mathbf{a}^T\mathrm{H}\mathbf{g}\right) = 1, \tag{17}$$

which can be formulated equivalently as the linear constraint:

$$\mathbf{a}^T\mathrm{H}\mathbf{g} \ge 1 \quad \forall \mathbf{a} : a_D = 1. \tag{18}$$

Substituting (18) into (16) yields:

$$\mathbf{g}_{PD} = \arg\min_g\left(\mathbf{g}^T\mathbf{g}\right), \; s.t. \; \mathbf{a}^T\mathrm{H}\mathbf{g} \ge 1 \;\; \forall \mathbf{a} : a_D = 1. \tag{19}$$

The problem (19) is a quadratic program (QP), with $2^{2L+K}$ linear constraints, which minimizes the worst-case sequence error probability, given the precise channel convolution matrix (5), and thus the PD equalizer minimizes the BER when the probability is sufficiently low and the eye diagram is open at the equalizer output. This QP is equivalent to the noise limited PD optimization problem presented in [4]. In the following we will develop a robust PD equalizer based on (19).

## 4. ROBUST EQUALIZATION

In this section we formulate a robust version of the PD equalizer, by explicitly incorporating channel uncertainties into the optimization problem (19). In many cases the channel may not be known precisely, but the i-th tap may be given by:

$$h_i + \varepsilon_i \Delta h_i \tag{20}$$

where $h_i$ represents the nominal channel coefficients, $\Delta h_i$ are the absolute maximum distortions on every channel coefficient and $\varepsilon_i$ is the i-th element of the vector $\boldsymbol{\varepsilon}$, which is unknown and bounded by $\|\boldsymbol{\varepsilon}\| \leq 1$. In practical cases the bounds $\Delta h_i$ will be derived from the specifics of the equalization problem at hand, for example: a time varying channel with a known bound on the varying rate or channel impulse response coefficients described with limited precision. Another example is the case in which the channel impulse response is estimated based on training sequences. In this case, the channel estimation error can be modeled, as in [9], and the $\Delta h_i$, can be chosen to be proportional to the standard deviation of the estimator. Defining a distortion matrix for every channel coefficient:

$$\Delta \mathbf{H_i} = \begin{bmatrix} 0 & & \\ \vdots & \ddots & \\ \Delta h_i & \ddots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ddots & \Delta h_i \\ & \ddots & \vdots \\ & & 0 \end{bmatrix}, \tag{21}$$

we can rewrite (20) in matrix form, as:

$$\mathbf{H} = \mathbf{H_0} + \varepsilon_0 \Delta \mathbf{H_0} + \cdots + \varepsilon_\mathbf{K} \Delta \mathbf{H_K}\ , \tag{22}$$

so that, $\mathbf{H}$ belongs to the set

$$\mathbf{S} \triangleq \left\{ \mathbf{H} \middle| \mathbf{H} = \mathbf{H_0} + \varepsilon_0 \Delta \mathbf{H_0} + \cdots + \varepsilon_\mathbf{K} \Delta \mathbf{H_K}, \|\boldsymbol{\varepsilon}\| \leq 1 \right\}. \tag{23}$$

In view of the above channel uncertainty model, the standard MMSE (7), ZF (9) and the PD equalizers (19) may not achieve the designated optimality. In order to account for the given channel uncertainty, we define the robust PD (RPD) equalizer as a solution to the problem of minimizing the PD criterion (19) for any $\mathrm{H} \in S$. Hence, the RPD criterion can be stated as:

$$\mathbf{g_{RPD}} = \arg \begin{cases} \min\limits_{g} \left\{ \mathbf{g}^T \mathbf{g} \right\} \\ s.t.\ \mathbf{a}^T \mathbf{H} \mathbf{g} \geq 0\ \forall \mathrm{H} \in S, \forall \mathbf{a}:\ a_D = 1 \end{cases} . \tag{24}$$

The above RPD equalizer is designed to minimize the worst-case SINR over every possible transmitted sequence and over any $\mathrm{H} \in S$, and thus, for sufficiently low error probability, the RPD equalizer will minimize the BER for $\mathrm{H} \in S$. Defining the channel distortion output vector as:

$$\mathbf{v}(\mathbf{a}, \mathbf{g}, \Delta \mathbf{H}) = \left[ \mathbf{a}^T \Delta \mathbf{H}_0 \mathbf{g}, \ldots, \mathbf{a}^T \Delta \mathbf{H}_\mathbf{K} \mathbf{g} \right]^T, \tag{25}$$

we can rewrite (24) as:

$$\mathbf{g}_{RPD} = \arg \begin{cases} \min\limits_{\mathbf{g}} \left\{ \mathbf{g}^\mathbf{T} \mathbf{g} \right\} \\ s.t.\ \mathbf{a}^\mathbf{T} \mathrm{H}_0^\mathrm{T} \mathbf{g} + \boldsymbol{\varepsilon}^\mathbf{T} \mathbf{v}(\mathbf{a}, \mathbf{g}, \Delta \mathrm{H}) \geq 1,\ \ \forall \mathrm{H} \in S\ \ \forall \mathbf{a}:\ a_i = 1 \end{cases}, \tag{26}$$

where the minimization is over every $\mathrm{H} \in S$ and for every possible transmitted sequence of length 2L+K+1. From the Cauchy-Schwarz inequality, we have

$$\boldsymbol{\varepsilon}^\mathbf{T} \mathbf{v}(\mathbf{a}, \mathbf{g}, \Delta \mathbf{H}) \geq -\|\boldsymbol{\varepsilon}\| \cdot \|\mathbf{v}(\mathbf{a}, \mathbf{g}, \Delta \mathbf{H})\| \geq -\|\mathbf{v}(\mathbf{a}, \mathbf{g}, \Delta \mathbf{H})\|. \tag{27}$$

Furthermore, it is clear that, for some vector $\boldsymbol{\varepsilon}$, such that $\|\boldsymbol{\varepsilon}\| \leq 1$, the equality holds, i.e.:

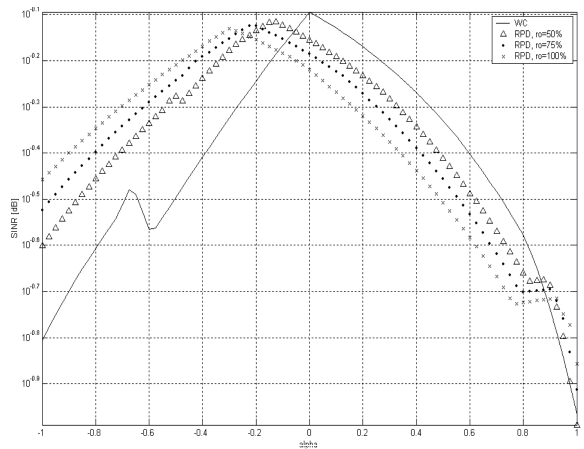


**Figure 1: Worst-case SINR performance for different robust designs.**

$$\min_{\mathbf{a}, \mathbf{g}, \boldsymbol{\varepsilon}} \left\{ \boldsymbol{\varepsilon}^\mathbf{T} \mathbf{v}(\mathbf{a}, \mathbf{g}, \Delta \mathbf{H}) \right\} = \min_{\mathbf{a}, \mathbf{g}} \left\{ -\|\mathbf{v}(\mathbf{a}, \mathbf{g}, \Delta \mathbf{H})\| \right\}, \tag{28}$$

hence, substituting the explicit form of the norm (27) into (26), yields the final optimization problem:

$$\mathbf{g}_{RPD} = \arg \begin{cases} \min\limits_{\mathbf{g}} \left\{ \mathbf{g}^\mathrm{T} \mathbf{g} \right\} \\ s.t.\ \mathbf{a}^\mathrm{T} \mathrm{H}_0^\mathrm{T} \mathbf{g}\ -1 \geq \sqrt{\sum\limits_{k=0}^{K} \left( \mathbf{a}^\mathrm{T} \Delta \mathrm{H}_k \mathbf{g} \right)^2},\ \forall \mathrm{H} \in S, \forall \mathbf{a}:\ a_i = 1 \end{cases} .\tag{29}$$

The above constraint is of the form $f_1(g) \geq \|\mathbf{f}_2(g)\|$, where the scalar $f_1(g)$ and the vector $\mathbf{f}_2(g)$ depend affinely on the optimization variables $\mathbf{g}$. Such inequalities define a convex set, which is called a second order cone (SOC). Thus, the SOC program in (29) can be solved efficiently using standard optimization packages, i.e., [8].

The number of constraints, in (29), is proportional to the number of different transmitted sequences with the length of the effective channel, i.e., the length of the convolution of the channel and equalizer, 2L+K+1. A solution to (29), $\mathbf{g}_{RPD}$, will satisfy the constraint (29) for every channel instance in the uncertainty region, $\mathrm{H} \in S$. Criterion (29) is a generalization of the standard PD criterion. In the case that there exists only one transmission channel, (29) reduces to the formulation of noise limited PD, [4]. Furthermore, for the single channel case, it is clear that any $\mathbf{g}_{RPD}$ is optimal in the sense of the original PD criterion [1].

Note, that if a single channel instance, from the set $\mathbf{S}$, will lead to a closed eye at the equalizer output, then the above SOC may not result in a feasible problem.

## 4. SIMULATION

In this section, we provide simulation results demonstrating the performance of the RPD equalizer. We have simulated different channels with different uncertainty parameters and measured the average error probability over the channel uncertainty set.

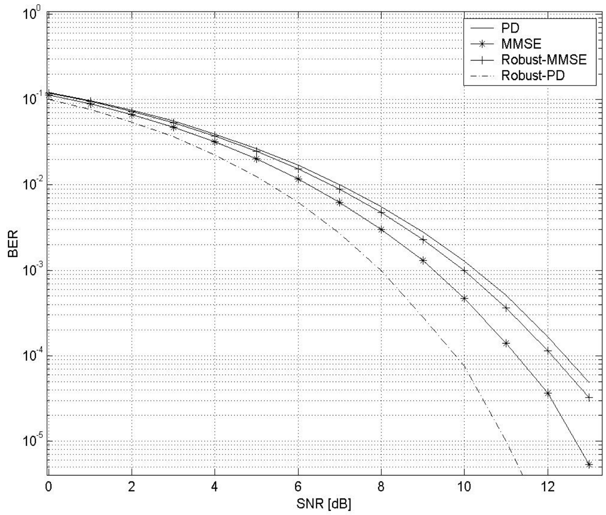


**Figure 2: BER performance for the worst-case channel,** $\alpha = 1$.

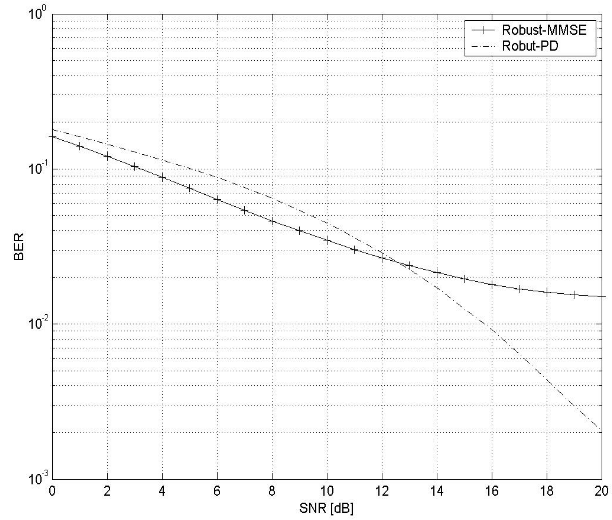


**Figure 3: Average BER performance, robust MMSE vs. robust PD.**

In all of our simulations the RPD equalizer outperformed the standard PD, MMSE and the robust MMSE [5] equalizers for sufficiently low error probability.

It is important to note, that in the case of channels with severe ISI, a linear equalizer is limited and does not attain these probabilities. But, in the cases where it did attain them, the RPD equalizer overall performance was superior. We now present three illustrative examples.

As a first example, we consider a case where the propagation channel, with 3 taps, is estimated with a given bound on the estimation error:

$$\mathrm{h} = (h_0, h_1, h_2) + \alpha(\Delta h_0, \Delta h_1, \Delta h_2), \qquad (30)$$

where $(h_0, h_1, h_2) = (1.0, 0.4, 0.15)$, $(\Delta h_0, \Delta h_1, \Delta h_2) = (0, 0.4, 0.25)$, and $\alpha \in [-1,1]$. We compare the worst-case sequence SINR performance, as a function of $\alpha$, for the PD and the RPD equalizers, with 4 taps. We repeat the calculation for different uncertainty regions scenario when we account for 50%, 75% and 100% of the given uncertainty region. For example, ro=50% in Fig. 1, means that in the design of the RPD, see (29), we used $0.5\Delta\mathrm{H}$. The results are provided in Fig. 1. Observing the performance of the standard PD equalizer, it is obvious that the performance degrades significantly as the propagation channel changes from the nominal value at $\alpha = 0$. The worst-case SINR of the worst-case channel is at $\alpha = 1$ and $\alpha = -1$. In the case of $\alpha = 1$, the robust counterparts succeed to maintain a lower amount of degradation

In Fig. 2, we compare the BER of the PD, MMSE, RPD and robust MMSE [5] equalizers. The RPD equalizer achieved an improvement of 3dB and 2dB from the standard PD equalizer and robust MMSE equalizer, respectively.

In Fig. 3, we compare the average BER performance of the RPD equalizer to the robust MMSE equalizer, with 4 taps. In the case that the channel is given by $(h_0, h_1, h_2) = (1.0, 0.8, 0.8)$, $(\Delta h_0, \Delta h_1, \Delta h_2) = (0, 0.1, 0.1)$ and $\alpha \in [-1,1]$. The results show that above 13dB, the RPD equalizer outperforms the robust MMSE equalizer significantly.

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