# RECONSTRUCTION OF NONUNIFORMLY SAMPLED PERIODIC SIGNALS: ALGORITHMS AND STABILITY ANALYSIS

*Evgeny Margolis and Yonina C. Eldar*

Department of Electrical Engineering
Technion – Israel Institute of Technology
Haifa, 32000, Israel

## ABSTRACT

This paper introduces two new algorithms for perfect reconstruction of a periodic bandlimited signal from its nonuniform samples. We analyze the advantages and disadvantages of each method and discuss their properties. Based on the theory of frames, we also analyze the stability of the algorithms. Some special structures of the sampling points are investigated and we show that uniform sampling results in the most stable and simple reconstruction algorithm. We also provide experimental evidence to support our theoretical results.

## 1. INTRODUCTION

The most common sampling used in the context of DSP is uniform sampling. However, the data can not always be sampled uniformly. In [1], Yao and Thomas proposed a formula for reconstruction of a non-periodic bandlimited signal from arbitrary spaced samples. There are several extensions of this theorem for special distributions of sampling points, among which are jittered sampling, recurrent nonuniform sampling, *etc.* [2].

Numerical implementation of these reconstruction methods on a digital computer is not possible due to the infinite dimension of the problem, *i.e.*, there is an infinite number of sampling points and the reconstruction functions typically have infinite length. Moreover, in most practical applications we are given only a finite number of samples, which makes a perfect reconstruction of the signal impossible. Any finite length signal can be periodically extended across the boundaries. Assumption that the reconstructed signal is periodic and bandlimited provides a simple and appropriate way to handle the problem of reconstruction of finite dimensional signals.

In this paper, we consider the problem of reconstructing a periodic bandlimited signal from nonuniform samples. We provide two reconstruction algorithms. The first, presented in Section 3, is using a set of reconstruction functions which forms a basis. The second, presented in Section 4, is using a set of overcomplete functions that span the appropriate space. For both algorithms we analyze the stability in two important cases: Uniform and recurrent nonuniform. We discuss advantages and disadvantages of each method, in particular the facts that the second is more stable and the first is consistent. Finally, the theoretical results are confirmed by simulations, which we present in Section 5.

## 2. PRELIMINARY NOTIONS

In this work, we consider the problem of reconstructing a periodic bandlimited signal $x(t)$ from its nonuniform samples. A real periodic signal $x(t)$, with period $T$, has a Fourier series representation $x(t) = \sum_{n=-\infty}^{\infty} c_n \exp\{j2\pi nt/T\}$ and Fourier transform

$$X(\omega) = \sum_{k=-\infty}^{\infty} c_k \delta\left(\omega - \frac{2\pi k}{T}\right), \qquad (1)$$

where $\delta(\omega)$ is the Dirac delta function. A $T$-periodic signal $x(t)$ is said to be *bandlimited* to $2\pi K/T$ if $c_k = 0$ for $|k| > K$. Such signals are also known as trigonometric polynomials of degree $K$. We will denote the space of $T$-periodic signals bandlimited to $2\pi K/T$ by $V_K$ and we will say that such signals are $K$-*bandlimited*. The dimension of the space $V_K$ is $M = 2K + 1$.

Our approach to reconstructing a periodic $K$-bandlimited signal $x(t)$ from its $N$ nonuniform samples $x(t_i)$ is to represent it as a linear combination of functions $\varphi_i(t)$, *i.e.*, $x(t) = \sum_{i=0}^{N-1} x(t_i)\varphi_i(t)$. These functions $\varphi_i(t)$ can either be linearly independent, in which case they form a basis for their span, or they can be linearly dependent, in which case they form a frame for their span. The set $\{\varphi_i(t)\}_{i=1}^{N}$ constitutes a frame for the space $V_K$ if there exist constants $A > 0$ and $B < \infty$ such that for all $x(t) \in V_K$ [3]

$$A\|x(t)\|^2 \le \sum_{i=1}^{N} |\langle x(t), \varphi_i(t)\rangle|^2 \le B\|x(t)\|^2, \qquad (2)$$

where $\langle x(t), y(t)\rangle = \frac{1}{T} \int_0^T x(t)y^*(t)dt$, $\|x(t)\|^2 = \langle x(t), x(t)\rangle$, and $y^*(t)$ is a complex conjugate of $y(t)$. The constants $A$ and $B$ are called the *frame bounds* and $r = N/M$ is the redundancy. If $N = M$, then the set $\{\varphi_i(t)\}$ is a *basis*. If the two frame bounds are equal, $A = B$, then the frame is called a *tight frame*. If in addition $N = M$, then the set is an *orthogonal basis*. The tightest possible bounds $A$ and $B$ in (2) can be found as the smallest and largest nonzero eigenvalues of the *frame correlation matrix $R$*, whose entries are given as

$$R_{ij} = \langle \varphi_i(t), \varphi_j(t)\rangle. \qquad (3)$$

One of the most important criteria of the reconstruction algorithm is its stability, namely the affect of a small perturbation of the samples on the reconstructed signal. As described in [4], the *condition number* of the reconstruction algorithm, which is given by the ratio

$$\kappa = B/A, \qquad (4)$$

provides an indicator of the stability and overall robustness of the reconstruction algorithm. The optimal situation is obviously $\kappa = 1$, which holds in the case of an orthogonal basis and a tight frame.

We now present a reconstructing algorithm, where set of reconstruction functions constitutes a basis. Then using that results, in Section 4 we develop a reconstruction based on frames.

## 3. RECONSTRUCTION WITH BASES

The problem of reconstructing a periodic $K$-bandlimited signal from uniform samples was considered in [5]. Reconstructing a periodic bandlimited signal from nonuniform samples is considerably more complicated. When the number of sampling points $N$ is odd and it satisfies $N \geq 2K + 1$, the reconstruction can be obtained using the Lagrange interpolation formula for trigonometric polynomials. For any even $N$ greater than $2K + 1$, Lagrange interpolation for exponential polynomials results in a complex valued interpolation function [6]. In Theorem 1 below we show that reconstruction can be obtained using real valued functions that are simpler than those derived in [6].

**Theorem 1.** *Let $x(t)$ be a $T$-periodic signal bandlimited to $2\pi K/T$. Then $x(t)$ can be perfectly reconstructed from its $N \geqslant 2K + 1$ nonuniformly spaced samples $x(t_p)$ as*

$$x(t) = \sum_{p=0}^{N-1} x(t_p) h_p(t), \qquad (5)$$

*where*

$$h_p(t) = \begin{cases} \prod_{\substack{q=0 \\ q \neq p}}^{N-1} \frac{\sin(\pi(t-t_q)/T)}{\sin(\pi(t_p-t_q)/T)}, & N \text{ odd}; \\ \cos\left(\frac{\pi(t-t_p)}{T}\right) \prod_{\substack{q=0 \\ q \neq p}}^{N-1} \frac{\sin(\pi(t-t_q)/T)}{\sin(\pi(t_p-t_q)/T)}, & N \text{ even}. \end{cases}$$
$$(6)$$

The proof of Theorem 1 follows from Yen's formula [2] for signal reconstruction from recurrent nonuniform samples, and from the fact that $x(nT + t_p) = x(t_p)$.

We can show that the functions $\{h_p(t)\}_{p=0}^{N-1}$ are linearly independent, *i.e.*, they form a basis. For $N$ odd they constitute a basis for the space $V_{(N-1)/2}$. In the case of $N$ even, we can show that $\{h_p(t)\}_{p=0}^{N-1}$ is a basis for the space $V_{(N-2)/2} \bigcup \sin(\pi(Nt - \sigma_t)/T)$, where $\sigma_t = \sum_{p=0}^{N-1} t_p$.

We can immediately verify that the reconstruction functions $\{h_p(t)\}_{p=0}^{N-1}$ of Theorem 1 have the *interpolation property*, namely

$$h_p(t_k) = \begin{cases} 1, & k = p, \\ 0, & k \neq p, \end{cases} \quad k, p = 0, 1, \dots, N-1. \quad (7)$$

If $x(t)$ is not bandlimited, then the reconstruction $\widehat{x}(t)$ given by Theorem 1 is not equal to $x(t)$. Nonetheless, the interpolation property (7) guaranties consistent reconstruction of the signal $x(t)$, *i.e.*, $\widehat{x}(t_p) = x(t_p)$. Consistency of the reconstruction algorithm is an important property for many signal processing applications [4].

We now consider two special cases of Theorem 1: Uniform sampling and recurrent nonuniform sampling.

### 3.1. Uniform Sampling

The most popular form of sampling used in the context of DSP is uniform sampling. In this case the set of reconstruction functions is given by [7]

$$h_p(t) = \begin{cases} \frac{\sin(N\pi(t-t_p)/T)}{N\sin(\pi(t-t_p)/T)}, & N \text{ odd}; \\ \cos\left(\frac{\pi(t-t_p)}{T}\right) \frac{\sin(N\pi(t-t_p)/T)}{N\sin(\pi(t-t_p)/T)}, & N \text{ even}. \end{cases} \quad (8)$$

For $N$ odd we can show that correlation matrix $R$ of the set $\{h_p(t)\}$ is equal to $1/N \mathbf{I}_{N \times N}$. Therefore, for $N$ odd the set

$\{h_p(t)\}$ constitutes an orthogonal basis, and its condition number is $\kappa = 1$, which is the lowest possible value for $\kappa$. Due to the low condition number, the set $\{h_p(t)\}$ of (8) provides stable reconstruction in the presence of noise.

For the case of $N$ even, the correlation matrix $R$ of the set $\{h_p(t)\}$ is a *circulant matrix* of the form

$$R = \begin{pmatrix} a & b & -b & b & \cdots & b \\ b & a & b & -b & \cdots & -b \\ \vdots & \vdots & & & \ddots & \end{pmatrix}, \qquad (9)$$

where $b = 1/(2N^2)$ and $a = 1/N - b$. Calculating the $N$ eigenvalues $\{\lambda_m\}_{m=0}^{N-1}$ of the circulant matrix as the discrete Fourier transform of its first row, we show that

$$\kappa = \frac{\max(\lambda_m)}{\min(\lambda_m)} = \frac{a+b}{a-(N-1)b} = 2. \qquad (10)$$

From (10) we conclude that the set of functions $\{h_p(t)\}$ is not an orthogonal basis, for which $\kappa = 1$. We also observe that $\kappa$ does not depend on $N$ and $T$, thus the set never converges to an orthogonal basis. As a result, reconstruction from an even number $N$ of uniform samples is less stable compared to $N$ odd, in which case the set $\{h_p(t)\}$ is orthogonal.

### 3.2. Recurrent Nonuniform Sampling

In this section, we consider the case of recurrent nonuniform sampling. In this form of sampling, the sampling points are divided into groups of $N_r$ nonuniformly spaced points. The groups have a recurrent period, which is denoted by $T_r$. One group of nonuniform samples repeats itself $M_r$ times along the $T$-periodic signal $x(t)$, where $M_r T_r = T$. Denoting the points in the first recurrent group by $t_r$, $r = 0, 1, \dots N_r - 1$, the complete set $\{t_p\}_{p=0}^{N-1}$ of sampling points in one period $T$ is

$$t_p = t_{p \bmod N_r} + \left\lfloor \frac{p}{N_r} \right\rfloor T_r, \quad p = 0, 1, \dots N-1, \qquad (11)$$

where $\lfloor \cdot \rfloor$ denotes the floor operator, which rounds down to the nearest integer. Substituting (11) into (6), we have

$$h_p(t) = \begin{cases} b_p \frac{\prod_{q=0}^{N_r-1} \sin(M_r \pi(t-t_q)/T)}{\sin(\pi(t-t_p)/T)}, & N \text{ odd}; \\ b_p \cos(\pi(t-t_p)/T) \frac{\prod_{q=0}^{N_r-1} \sin(M_r \pi(t-t_q)/T)}{\sin(\pi(t-t_p)/T)}, & N \text{ even}, \end{cases}$$
$$(12)$$

where

$$b_p = \frac{1}{M_r \prod_{q=0, q \neq p}^{N_r-1} \sin(M_r \pi(t_p - t_q)/T)}. \qquad (13)$$

An efficient implementation of (12) using a bank of continuous-time LTI filters was developed in [7]. The correlation matrix $R$ of (12) can be shown to be a *block circulant matrix* of the form

$$R = \begin{pmatrix} A_0 & A_1 & A_2 & \cdots & A_{M_r-1} \\ A_{M_r-1} & A_0 & A_1 & \cdots & A_{M_r-2} \\ \vdots & \vdots & & \ddots & \end{pmatrix}, \qquad (14)$$

where the submatrices $A_r$, $r = 0, \dots, M_r - 1$ are square $N_r \times N_r$ matrices. To compute the eigenvalues of $R$ we define the *discrete Fourier components* of $R$ as

$$\widehat{A}_k = \sum_{r=0}^{M_r-1} W^{kr} A_r, \quad k = 0, \dots, M_r - 1, \qquad (15)$$

where $W = e^{-j\frac{2\pi}{M_r}}$. Let $\{\lambda_{k,i}\}_{i=0}^{N_r-1}$ be the $N_r$ eigenvalues of $\widehat{A}_k$ for every $k = 0, \ldots, M_r - 1$. It was shown in [8] that the eigenvalues of an Hermitian block circulant matrix are the eigenvalues of the discrete Fourier components. Therefore, the eigenvalues of the matrix $R$ of (14) are given by

$$\lambda(R) = \lambda_{k,i}, \quad i = 0, \ldots, N_r - 1, \quad k = 0, \ldots, M_r - 1. \quad (16)$$

The last result (16), significantly simplifies the calculation of the condition number $\kappa$ of the reconstruction algorithm in the case of recurrent nonuniform samples. Instead of computing $N$ eigenvalues of an $N \times N$ matrix $R$, which may be very large, we compute $N_r$ eigenvalues of the $M_r$ matrices $\widehat{A}_k$ defined by (15), where typically $N_r << N$. In Section 4.2, we will see an example of the value of $\kappa$ for a case of recurrent nonuniform sampling.

In the next section, using the results of this section we present a reconstruction method based on frames.

## 4. FRAME BASED RECONSTRUCTION

Theorem 1 provides perfect reconstruction of periodic bandlimited signals from nonuniform redundant ($N > 2K + 1$) and nonredundant ($N = 2K + 1$) samples. Given the set of $N$ sampling points, Theorem 1 generates the same set of $N$ reconstruction functions for any $K$-bandlimited signal, where $K = 0, 1, \ldots, \lfloor (N-1)/2 \rfloor$. The fact that for $N > 2K + 1$ the signal is oversampled is not taken into account in the reconstruction process.

In the oversampled case, if the samples are corrupted by noise, applying a low pass filter of support $[-2\pi K/T, 2\pi K/T]$ on the reconstructed signal can reduce the average power of the noise. We denote this low pass filtering as an operator $P_K$, which zeros all harmonics higher than $K$ of the periodic signal $x(t)$. We can immediately verify that the operator $P_K$ is an orthogonal projector onto the space $V_K$, namely it satisfies

$$\begin{aligned} P_K x(t) &= x(t), \quad x(t) \in V_K; \\ P_K x(t) &= 0, \quad\quad x(t) \in V_K^\perp, \end{aligned} \quad (17)$$

where $V_K^\perp$ is the space of functions orthogonal to $V_K$, *i.e.*, the space of $T$-periodic functions with $c_k = 0$ for $|k| \leq K$. Applying $P_K$ to the reconstructed signal of (5) results in

$$x(t) = P_K \sum_{p=0}^{N-1} x(t_p) h_p(t) = \sum_{p=0}^{N-1} x(t_p) P_K h_p(t), \quad (18)$$

which is equivalent to a reconstruction with the set of functions $\{P_K h_p(t)\}$. To determine the properties of this set, we rely on the following proposition.

**Proposition 1.** *Let $\{\varphi_i(t)\}_{i=1}^N$ be a basis for a space $W$ with frame bounds $A_W$ and $B_W$, and let $P$ denote the orthogonal projection of $W$ onto a closed subspace $V$. Then $\{P\varphi_i(t)\}_{i=1}^N$ is a frame for $V$ with frame bounds $A_V \geq A_W$ and $B_V \leq B_W$.*

Proposition 1 leads directly to the conclusion that the set of functions $\{P_K h_p(t)\}$ constitutes a frame for the space $V_K$ with redundancy ratio $r = N/(2K + 1)$. Computing the functions $\{P_K h_p(t)\}$ in (18) leads to a new reconstruction theorem.

**Theorem 2.** *The problem of reconstructing a $T$-periodic $K$- bandlimited signal from arbitrary spaced samples, considered in Theorem 1, can be solved by (5), with*

$$h_p(t) = \frac{\alpha_{p0}}{2} + \sum_{k=1}^K (\alpha_{pk} \cos(2\pi kt/T) - \beta_{pk} \sin(2\pi kt/T)). \quad (19)$$

*For $N$ odd*

$$\alpha_{pk} = \frac{a_p(-1)^k}{2^{N-2}} \sum_{\varphi \in G_{pk}} \cos(\pi\varphi/T),$$

$$\beta_{pk} = \frac{a_p(-1)^k}{2^{N-2}} \sum_{\varphi \in G_{pk}} \sin(\pi\varphi/T), \quad (20)$$

*where $G_{pk}$ is the set of all possible sums of values $\{t_q\}_{q=0,q\neq p}^{N-1}$, when $(N-1)/2 + k$ of them chosen with negative sign and $(N-1)/2 - k$ are positive.*

*For $N$ even*

$$\alpha_{pk} = \frac{a_p(-1)^k}{2^{N-1}} \left( \sum_{\varphi \in G_{pk}^+} \sin(\pi\varphi/T) - \sum_{\varphi \in G_{pk}^-} \sin(\pi\varphi/T) \right),$$

$$\beta_{pk} = \frac{a_p(-1)^k}{2^{N-1}} \left( \sum_{\varphi \in G_{pk}^-} \cos(\pi\varphi/T) - \sum_{\varphi \in G_{pk}^+} \cos(\pi\varphi/T) \right) \quad (21)$$

*where the set $\{G_{pk}^+ \oplus G_{pk}^-\}$ consists of all possible sums of values $\{t_q\}_{q=0}^{N-1}$, when $N/2 + k$ of them chosen with negative sign and $N/2 - k$ are positive. The value of $t_p$ appears with positive and negative signs in $G_{pk}^+$ and $G_{pk}^-$, respectively.*

In the ideal case, where $x(t) \in V_K$ and its samples are not corrupted ny noise, the reconstruction functions (6) and (19) both, lead to the perfect reconstruction of the signal. However, when $x(t)$ is not truly bandlimited or its samples $\{x(t_p)\}$ are corrupted by noise, these two methods lead to different reconstructions. It follows from Proposition 1 that the frame bounds $A$, $B$ of the set (19) are tighter than the bounds of the set (6). As a result, the condition number $\kappa$ of the frame (19) is smaller than $\kappa$ of (6). This fact makes the frame based method of Theorem 2 more robust than the algorithm of Theorem 1. However, interpolation property (7) no longer holds when using (19), *i.e.*, the reconstructed signal $\widehat{x}(t)$ does not necessarily satisfy $\widehat{x}(t_p) = x(t_p)$.
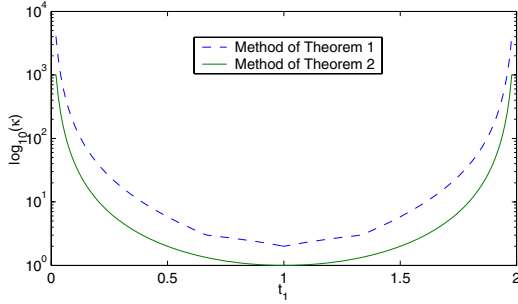
### 4.1. Uniform Sampling

In the case of uniform samples, projecting the set (8) onto the space $V_K$, we obtain

$$h_p(t) = \frac{\sin(\pi(2K+1)(t - t_p)/T)}{N \sin(\pi(t - t_p)/T)}. \quad (22)$$

We can show that the functions $\{h_p(t)\}$ of (22) constitute a tight frame for $V_K$, so that $\kappa = 1$. Comparing it to the set (8), we notice an improvement in condition number in the case of an even number of sampling points, where it reduces from $\kappa = 2$ to $\kappa = 1$.

### 4.2. Recurrent Nonuniform Sampling

For the case of recurrent nonuniform samples, we do not give an explicit formula for the set of reconstruction functions due to complexity of representation, but we do provide a countable example.

**Fig. 1**. Comparison between the condition number $\kappa$ of two reconstruction methods, as a function of $t_1$.

We now consider a set of $N = 10$ recurrent nonuniform samples, with $N_r = 2$, $M_r = 5$, and $T = 10$. We fix the value of $t_0 = 0$ and allow $t_1$ to change in the range $[0, 2]$. The signal $x(t)$ lies in the space $V_2$.

The behavior of the condition number as a function of $t_1$ is shown in Fig. 1, where the eigenvalues of the block circulant correlation matrix $R$ were calculated using the method presented in Section 3.2. As predicted by Proposition 1, the condition number of the frame method is significantly lower than $\kappa$ of the set of reconstruction functions of Theorem 1. We observe, that in the neighborhood of $t_1 = 1$ the condition number of the frame based method is very low and it achieves the minimal value for $t_1 = 1$, where the recurrent nonuniform sampling set becomes uniform and the set of reconstruction functions constitutes a tight frame. For the reconstruction method of Theorem 1, since $N$ is even the lowest value of $\kappa$ for a uniform sampling set is 2 (10).

## 5. SIMULATION RESULTS

In this section, we present some numerical results and simulations. We created a 10-periodic 4-bandlimited signal $x(t)$, which belongs to the 9-dimensional space $V_4$. We then consider three sets of 18 sampling points $\{t_p\}$: nonuniform, uniform, and recurrent nonuniform with $N_r = 3$, where the nonuniform points were randomly chosen. Each set of samples $\{x(t_p)\}$ was perturbed by a randomly generated Gausian sequence $\{w_p\}$ with zero mean and variance 0.01. In Fig. 2, we plot the signal $x(t)$ and two reconstructions obtained with Theorem 1 and 2. Noisy samples $\{x(t_p) + w_p\}$ are marked by dots.

We define the indicator of the quality of the reconstruction as the signal to noise ratio (SNR), given by $10 \log_{10} \frac{\|x(t)\|^2}{\|x(t) - \hat{x}(t)\|^2}$. For the examples of Fig. 2, the SNR values are given in the Table 1.

**Table 1**. SNR of the Reconstructed Signals of Fig. 2

| Reconstruction Method | SNR [dB] | | |
|---|---|---|---|
| | Nonuniform | Uniform | Recurrent |
| Theorem 1 | 3.62 | 16.84 | 13.08 |
| Theorem 2 | 6.35 | 19.87 | 19.82 |

We can clearly see from Fig. 2 and Table 1 that the frame method of Theorem 2 results in a higher quality of reconstruction.
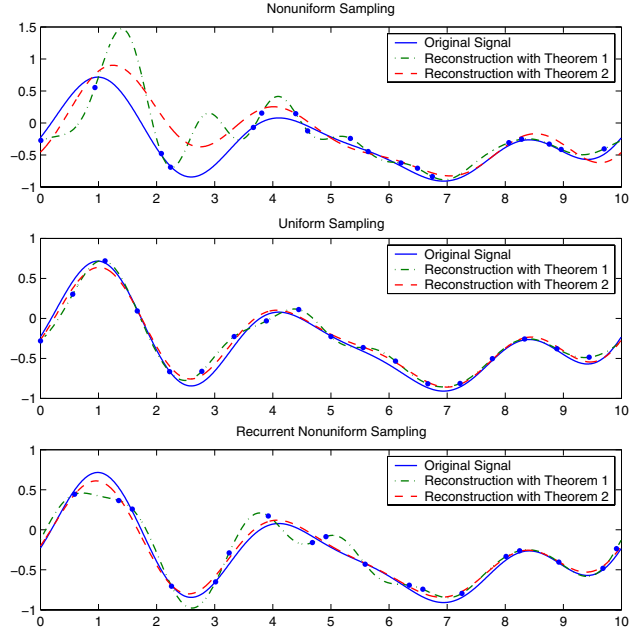


**Fig. 2**. Comparison between the reconstruction methods.

We also observe that there is the same error for uniform and recurrent nonuniform sampling sets, when the method of Theorem 2 is used for reconstruction. An important observation is that, owing to (7), the method of Theorem 1 provides consistent reconstruction of the signals. This property is very useful in many signal processing applications [4], *e.g.,* in interpolation theory.

## 6. REFERENCES

[1] K. Yao and J. B. Thomas, "On some stability and interpolatory properties of nonuniform sampling expansions," *IEEE Trans. Circuit Theory*, vol. CT-14, pp. 404–408, Dec. 1967.

[2] J. L. Yen, "On nonuniform sampling of bandwidth-limited signals," *IEEE Trans. Circuit Theory*, vol. CT-3, pp. 251–257, Dec. 1956.

[3] O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, 2002.

[4] Y. C. Eldar, "Sampling without input constraints: Consistent reconstruction in arbitrary spaces," to appear in *Sampling, Wavelets and Tomography*, A. Zayed and J. J. Benedetto, Ed.

[5] T. Schanze, "Sinc interpolation of discrete periodic signals," *IEEE Trans. Signal Processing*, vol. 43, pp. 1502–1503, June 1995.

[6] F. A. Marvasti, *A Unified Approach to Zero-Crossings and Nonuniform Sampling*, Illinois Inst. of Tech., Chicago, 1987.

[7] E. Margolis and Y. C. Eldar, "Filterbank reconstruction of periodic signals and sampling in polar coordinates," *Proceedings of the 2003 Workshop on Sampling Theory and Applications, SampTA'03*, May 2003.

[8] A. Beck, Y. C. Eldar, and A. Ben-Tal, "Minimax mean-squared error estimation of multichannel signals," to be submitted to *IEEE Trans. Signal Processing*.