FILTERBANK RECONSTRUCTION OF PERIODIC SIGNALS AND SAMPLING IN POLAR COORDINATES

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ABSTRACT

This paper introduces a new reconstruction formula for reconstructing a periodic bandlimited signal from its nonuniform samples, as well as filterbank interpretations of various sampling strategies, which lead to efficient interpolation and reconstruction methods. As an example of a potential application of these results, we apply it to the problem of reconstructing a two-dimensional bandlimited signal from its recurrent nonuniform samples in polar coordinates, and develop an efficient filterbank reconstruction with bandlimited LTI filters.

1. INTRODUCTION

Digital signal processing (DSP) and image processing theory rely on sampling a continuous-time (CT) signal to obtain a discrete-time (DT) representation of the signal. The most common form of sampling used in the context of DSP is uniform sampling. However, there are a variety of applications in which data is sampled in other ways, such as nonuniformly in time. The problem of reconstructing a signal from its nonuniform samples arises in a variety of fields such as medical imaging, geophysics and speech processing. In particular, the problem of signal reconstruction from its nonuniform frequency domain samples arises in computerized tomography and magnetic resonance imaging (MRI). Polar sampling strategies, such as linear spiral scan techniques [1], which are widely used in MRI, provide practical advantages in the context of medical imaging. To avoid artifacts in the reconstruction process, efficient interpolation methods from nonuniform samples in polar coordinates are required.

In this paper, we consider the problem of reconstruction from nonuniform samples, where the underlying signal is periodic in the one-dimensional (1-D) case, and periodic in one dimension in the two-dimensional (2-D) case. In practice, any signal with finite time (space) support can be represented as a periodic signal, an approach frequently encountered in image processing. In Section 2, we develop a new sampling theorem for reconstruction of a periodic signal from its nonuniform samples. We then apply this theorem to reconstruction of a periodic bandlimited signal from recurrent nonuniform samples, and in Section 3, we develop a filterbank (FB) interpretation of the reconstruction process. Since any function $f(r,\theta)$ given in polar coordinates is 2π-periodic in θ, these results can be applied to the reconstruction of 2-D signals from nonuniformly spaced samples in polar coordinates. As an example, in Section 4, we develop a FB interpretation of reconstruction of a bandlimited signals from recurrent nonuniform samples in polar coordinates.

2. NONUNIFORM SAMPLING OF PERIODIC BANDLIMITED SIGNALS

In this section, we consider the problem of reconstructing a periodic bandlimited signal from its nonuniform samples. A periodic signal $x(t)$, with period T, has a Fourier series representation $x(t)=\sum_{n=-\infty}^{\infty} c_n \exp(j2\pi nt/T)$ and Fourier transform

$$X(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt=\sum_{n=-\infty}^{\infty} c_n\delta\left(\omega-\frac{2\pi n}{T}\right), \tag{1}$$

where $\delta(\omega)$ is the Dirac delta function. A T-periodic signal $x(t)$ is said to be bandlimited to $2\pi K/T$ if $c_n=0$ for $|n|>K$. Theorem 1 below asserts that such a signal can be perfectly reconstructed from a finite number N of its arbitrary spaced samples, where $N\geqslant 2K+1$.

The problem of reconstructing a periodic bandlimited signal from $N=2K+1$ uniform samples was first considered by Cauchy [2], and later by Stark [3], Brown [4] and Schanze [5]. Reconstruction from any even number N of uniform samples was considered in [6] and [5]. As we show, these results are all special cases of Theorem 1. Reconstructing a periodic bandlimited signal from nonuniform samples is considerably more complicated. When $N=2K+1$, reconstruction can be obtained using the Lagrange interpolation formula for trigonometric polynomials [7]. For $N>2K+1$, Lagrange interpolation for exponential polynomials results in a complex valued interpolation function [8]. In Theorem 1 below we show that interpolation can be obtained using real valued functions that are simpler than those derived in [8].

Theorem 1 *Let $x(t)$ be a T-periodic signal bandlimited to $2\pi K/T$. Then $x(t)$ can be perfectly reconstructed from its $N\geqslant 2K+1$ nonuniformly spaced samples $x(t_p)$ as*

$$x(t)=\sum_{p=0}^{N-1} x(t_p)h_p(t), \tag{2}$$

where

$$h_p(t)=\begin{cases}\displaystyle\prod_{\substack{q=0\\q\neq p}}^{N-1}\frac{\sin(\pi(t-t_q)/T)}{\sin(\pi(t_p-t_q)/T)}, & N \textit{ odd};\\[2ex] \displaystyle\cos\left(\frac{\pi(t-t_p)}{T}\right)\prod_{\substack{q=0\\q\neq p}}^{N-1}\frac{\sin(\pi(t-t_q)/T)}{\sin(\pi(t_p-t_q)/T)}, & N \textit{ even}.\end{cases} \tag{3}$$

The proof of Theorem 1 follows from Yen's formula [10] for signal reconstruction from nonuniform recurrent samples, and the fact that $x(nT + t_p) = x(t_p)$.

The interpolation function $h_p(t)$ of Theorem 1 is periodic in T and has the interpolation property, namely

$$h_p(t_k) = \begin{cases} 1, & k = p \\ 0, & k \neq p \end{cases} \qquad k, p = 0, 1, \ldots, N-1, \tag{4}$$

which is very important in approximation theory. Specifically, if $x(t)$ is not precisely bandlimited, then the reconstruction $\widehat{x}(t)$ given by Theorem 1 may not be equal to $x(t)$. Nonetheless, the interpolation property (4) guaranties that $\widehat{x}(t_p) = x(t_p)$.

We now consider two special cases of Theorem 1: Uniform sampling and recurrent nonuniform sampling.

Uniform Sampling:
Suppose that the sampling points are $t_p = pT/N$, $p = 0, 1, \ldots N-1$, *i.e.,* uniformly spaced over one period T. To develop $h_p(t)$ for this case we first note that

$$\prod_{\substack{q=0 \\ q\neq p}}^{N-1} \frac{\sin(\pi(t-t_q)/T)}{\sin(\pi(t_p-t_q)/T)} = \frac{2}{N} D_N(t - t_p), \tag{5}$$

where $D_N(t)$ is the *Dirichlet kernel* [7], which can equivalently be expressed as

$$D_N(t) = \frac{N}{2} \prod_{k=1}^{N-1} \frac{\sin(\pi(t-t_k)/T)}{\sin(\pi t_k/T)} = \frac{\sin(N\pi t/T)}{2\sin(\pi t/T)}. \tag{6}$$

Substituting (5) and (6) into (3), we have

$$h_p(t) = \begin{cases} \frac{\sin(N\pi(t-t_p)/T)}{N\sin(\pi(t-t_p)/T)}, & N \text{ odd;} \\ \cos\left(\frac{\pi(t-t_p)}{T}\right) \frac{\sin(N\pi(t-t_p)/T)}{N\sin(\pi(t-t_p)/T)}, & N \text{ even,} \end{cases} \tag{7}$$

which is equal to the interpolation function derived in [3], [2], [5] and [4].

Recurrent Nonuniform Sampling:
We now consider the case in which the sampling points are defined by a group of N_t nonuniformly spaced points, which repeats itself M_t times with recurrent period T_t along the T-periodic signal, where $M_t T_t = T$. Denoting the points in one group by t_p, $p = 0, 1, \ldots N_t - 1$, the complete set of sampling points in one period T is

$$t_p + nT_t, \quad p = 0, 1, \ldots N_t - 1, \;\; n = 0, 1, \ldots M_t - 1. \tag{8}$$

Recurrent nonuniform samples can be regarded as a combination of N_t sequences of uniform samples with M_t points each, taken with interval T_t. An example of a sampling distribution for the case $N_t = 3$ and $M_t = 2$ is depicted in Fig. 1.

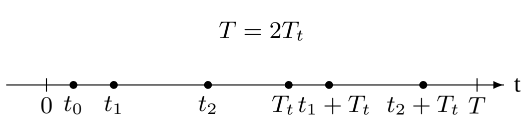

Fig. 1. Sampling distribution for $N_t = 3$ and $M_t = 2$.

From (8) and (3), the interpolation function in this case is

$$h_p(t) = \begin{cases} b_p \frac{\prod_{q=0}^{N_t-1} \sin(M_t\pi(t-t_q)/T)}{\sin(\pi(t-t_p)/T)}, & N \text{ odd;} \\ b_p \cos(\pi(t-t_p)/T) \frac{\prod_{q=0}^{N_t-1} \sin(M_t\pi(t-t_q)/T)}{\sin(\pi(t-t_p)/T)}, & N \text{ even,} \end{cases} \tag{9}$$

where

$$b_p = \frac{1}{M_t \prod_{q=0, q\neq p}^{N_t-1} \sin(M_t\pi(t_p - t_q)/T)}. \tag{10}$$

Direct reconstruction using (9) is computationally difficult. In the next section we develop an efficient implementation of (9) using a bank of CT LTI filters.

3. RECONSTRUCTION USING LTI FILTERS

In this section, we develop a CTFB interpretation of the reconstruction from uniform samples (7), and the reconstruction from nonuniform samples (9).

Uniform Sampling:
In this case the reconstruction (2) can be expressed as

$$x(t) = s(t) * h(t), \tag{11}$$

where

$$h(t) = \begin{cases} \frac{\sin(N\pi t/T)}{N\sin(\pi t/T)}, & N \text{ odd;} \\ \cos(\pi t/T) \frac{\sin(N\pi t/T)}{N\sin(\pi t/T)}, & N \text{ even,} \end{cases} \tag{12}$$

and $s(t)$ is an impulse train of samples,

$$s(t) = \sum_{p=0}^{N-1} x(t_p)\delta(t - t_p), \qquad t_p = \frac{pT}{N}. \tag{13}$$

From (11) it follows that $x(t)$ is obtained by filtering $s(t)$ with an LTI filter with impulse response $h(t)$ given by (12), and frequency response $H(\omega)$ given by

$$\begin{aligned} H(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin(N\pi t/T)}{N\sin(\pi t/T)} e^{-j\omega t} dt \\ &= \frac{1}{N} \sum_{n=-(N-1)/2}^{(N-1)/2} \delta\left(\omega - \frac{2\pi n}{T}\right), \end{aligned} \tag{14a}$$

for N odd, and

$$\begin{aligned} H(\omega) = &\frac{1}{N} \sum_{n=-(N-2)/2}^{(N-2)/2} \delta(\omega - \frac{2\pi n}{T}) \\ &+ \frac{1}{2N}\left(\delta(\omega - \frac{N\pi}{T}) + \delta(\omega + \frac{N\pi}{T})\right), \end{aligned} \tag{14b}$$

for N even.

Evidently, $H(\omega) = 0$ for $|\omega| > \pi(N-1)/T$ when N is odd, and $H(\omega) = 0$ for $|\omega| > \pi N/T$ when N is even, *i.e.,* the filters in both cases are bandlimited. The frequency response of $H(\omega)$ for the case $N = 10$ and $T = 2\pi$ is shown in Fig. 2.

Note, that the filter with frequency response depicted in Fig. 2, is used to reconstruct signals bandlimited to $2\pi K/T = 4$. The

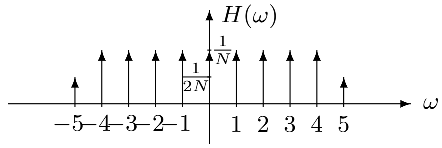

Fig. 2. Frequency response of the reconstruction filter $H(\omega)$ of (14b) for $N = 10$ (even) and $T = 2\pi$.

filter includes an unnecessary 5th harmonic, which does not contribute to the reconstruction process, when $x(t)$ is truly bandlimited to 4. In this case, if we replace this last harmonic by 0, then the impulse response of the resulting filter is

$$h(t) = \frac{\sin(\pi(2K+1)t/T)}{N\sin(\pi t/T)}, \tag{15}$$

which was derived in [6]. We may then use the interpolation functions $h_p(t) = h(t - t_p)$ defined by (15) to reconstruct a periodic bandlimited signal from an arbitrary number N of samples, where $N \geqslant 2K + 1$. The functions $\{h_p(t)\}$ constitute a tight frame [11] for the space of T-periodic signals bandlimited to $2\pi K/T$ with redundancy ratio $N/(2K + 1)$.

In the case in which $x(t)$ is bandlimited to $2\pi K/T$, the interpolation functions (15) and (12) lead to the same reconstruction. However, when $x(t)$ is not truly bandlimited, the interpolation functions will lead to different reconstructions. The interpolation function (12) has the desirable interpolation property, so that the reconstruction $\widehat{x}(t)$ satisfies $\widehat{x}(t_p) = x(t_p)$. This property no longer holds when using (15).

Recurrent Nonuniform Sampling:
We now show that the reconstruction formula (9) from recurrent nonuniform samples can be implemented using a CTFB. Combining (9) and (2), the reconstruction can be expressed as a sum of N_t convolutions,

$$x(t) = \sum_{p=0}^{N_t-1} s_p(t) * h_p(t), \tag{16}$$

where

$$h_p(t) = b_p\ \frac{\sin(M_t\pi t/T)}{\sin(\pi t/T)} \prod_{\substack{q=0\\ q\neq p}}^{N_t-1} \sin(M_t\pi(t+t_p-t_q)/T) \tag{17a}$$

for odd N,

$$h_p(t) = b_p\ \cos(\pi t/T)\frac{\sin(M_t\pi t/T)}{\sin(\pi t/T)} \prod_{\substack{q=0\\ q\neq p}}^{N_t-1} \sin(M_t\pi(t+t_p-t_q)/T) \tag{17b}$$

for even N, and $s_p(t)$ is an impulse train of samples,

$$s_p(t) = \sum_{n=0}^{M_t-1} x(nT_t + t_p)\delta(x - nT_t - t_p). \tag{18}$$

Equation (16) can be interpreted as a filterbank with N_t filters. Each uniform sequence of samples $s_p(t)$ formed according to (18) is filtered by a CT filter $H_p(\omega)$, with impulse response given by (17). Summing the outputs of the N_t filters results in the reconstructed signal $x(t)$.

To determine the frequency response $H_p(\omega)$ of the filter $h_p(t)$ of (17b) for an even number of samples, we note that $h_p(t)$ of (17b) can be expressed as

$$h_p(t) = b_p\ \cos(\pi t/T)\frac{\sin(M_t\pi t/T)}{\sin(\pi t/T)} \sum_{k=-N_t+1}^{N_t-1} c_k e^{jkM_t\pi t/T}, \tag{19}$$

where the complex coefficients c_k are the result of expanding the product of sines in (17b) into complex exponentials. As shown in (14b), the first term in (19) is a filter bandlimited to $\pi M_t/T$. The effect of the summation and multiplexing by the exponent is to create shifted and scaled versions of this bandlimited filter, so that the filter response $H_p(\omega)$ is bandlimited to $\pi M_t N_t/T$.

Applying the interpolation identity derived in [12], the CTFB, derived in this section, can be converted to a DTFB, which results in interpolation of the nonuniform samples to uniformly spaced samples.

In Fig. 3 we depict $H_2(\omega)$ for the case in which $T = 2\pi$, and the nonuniform samples are given by $t_0 = 0$, $t_1 = 0.087$, $t_2 = 0.227$ repeated with period $T_t = \pi/6$ so that $N_t = 3$ and $M_t = 12$. As we expect, the filter is bandlimited to $\pi M_t N_t/T = 12 \cdot 3/2 = 18$, since $H_2(\omega)$ is created by three (N_t) shifted and scaled versions of a filter bandlimited to $M_t/2 = 6$.

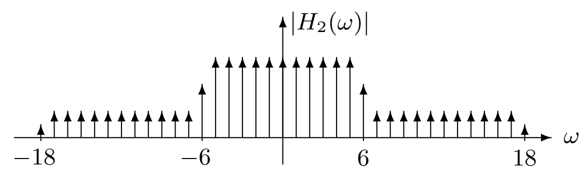

Fig. 3. Frequency response of $H_2(\omega)$.

4. RECONSTRUCTION IN POLAR COORDINATES

We now consider an application of Theorem 1 and the FB interpretation of Section 3, to the problem of reconstruction of 2-D signals from nonuniform samples in polar coordinates.

Butzer *et al.* [13] considered reconstruction from nonuniform samples in 2-D cartesian coordinates, in which the nonuniform sampling points all lie on straight lines parallel to the y axis, and the average sampling density in both coordinates is greater than the Nyquist rate. As we show, Butzer's interpolation theorem can be extended to nonuniform sampling in polar coordinates, in which nonuniform samples lie either on concentric circles or on radial lines, where these circles or lines are nonuniformly distributed [9].

We now consider the case of recurrent nonuniform sampling in polar coordinates, where we perform recurrent nonuniform sampling along each one of coordinates r and θ. In the radial direction we define a group of N_r samples repeated with period T_r, and in the azimuthal direction we define a group of N_θ samples with period T_θ. A sampling grid for recurrent nonuniform samples in polar coordinates with $M_\theta = 12$, $N_\theta = 3$ and $N_r = 3$ is depicted in Fig. 4. In this sampling scheme, samples on radial lines are always symmetric about $r = 0$ to provide radial symmetry of the grid. As can be seen from the figure, the nonuniform group of N_θ samples in the θ coordinate always has an even number of repetitions, *i.e.*, M_θ is even, so that the total number of samples $N = N_\theta M_\theta$ is also even.

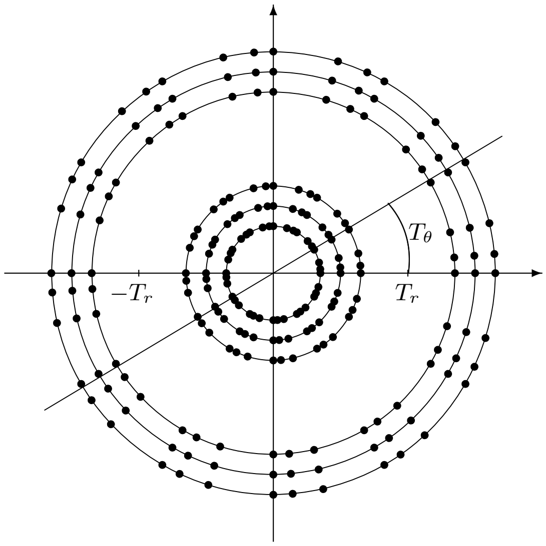

Fig. 4. Recurrent sampling in polar coordinates (r, θ) with $M_\theta = 12$, $N_\theta = 3$ and $N_r = 3$.

Extending Butzer's theorem to polar coordinates, it can be shown that any function $f(r, \theta)$ bandlimited to a circular disc of radius $\pi N_r/T_r$ and angularly bandlimited to $(N_\theta M_\theta - 2)/2$ can be perfectly reconstructed from the recurrent nonuniform samples defined above.

Since, any function $f(r, \theta)$ in polar coordinates is 2π-periodic in θ, interpolation from the azimuthal coordinate can be obtained using Theorem 1, with $T = 2\pi$. Reconstruction from the radial coordinate can be obtained using the known results on reconstruction from recurrent nonuniform samples in 1-D [12]. Using this approach, we can show that the reconstruction can be expressed as a sum of $N_r N_\theta$ convolutions,

$$f(r, \theta) = \sum_{p=0}^{N_r-1} \sum_{q=0}^{N_\theta-1} s_{pq}(r, \theta) * h_{pq}(r, \theta), \qquad (20)$$

where $s_{pq}(r, \theta)$ is a 2-D impulse train of samples,

$$s_{pq}(r, \theta) = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{M_\theta-1} \widetilde{f}(nT_r + r_p, mT_\theta + \theta_q) \cdot \delta(r - nT_r - r_p, \theta - mT_\theta - \theta_q). \qquad (21)$$

Here $\widetilde{f}(r, \theta)$ is an extension of $f(r, \theta)$ to new coordinates [14], with $-\infty < r < \infty$ and $0 \leqslant \theta < 2\pi$,

$$\widetilde{f}(r, \theta) = \begin{cases} f(r, \theta), & r \geqslant 0; \\ f(-r, \theta + \pi), & r < 0, \end{cases} \qquad (22)$$

and $h_{pq}(r, \theta) = h_p^r(r) h_q^\theta(\theta)$ is a separable 2-D filter, where $h_q^\theta(\theta)$ is given by (17b) with $M_t = M_\theta, T_t = T_\theta$ and $T = 2\pi$, and $h_p^r(r)$ follows from the results in [12]. From the discussion in Section 3, the filter $h_q^\theta(\theta)$ is bandlimited to $M_\theta N_\theta/2$, and from [12], the filter $h_p^r(r)$ is bandlimited to $\pi N_r/T_r$.

We note that in [9] Marvasti also considered the problem of reconstruction from nonuniform samples in polar coordinates. The interpolation function he developed involves complex-valued functions, and is therefore more complicated to implement. Furthermore, he does not develop an efficient interpolation method using LTI filters, as we do here.

5. CONCLUSION

We presented a formula for reconstruction of T-periodic signals bandlimited to $2\pi K/T$ from nonuniform samples, and developed an efficient FB interpretation of the reconstruction in the case of recurrent nonuniform sampling. We then applied these result to the reconstruction of 2-D signals from nonuniform samples in polar coordinates, and provided a CTFB interpretation of the reconstruction using bandlimited LTI filters.

6. REFERENCES

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