Unambiguous Quantum State Discrimination and the Equal Probability Measurement

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I. Introduction

In a quantum detection problem a transmitter conveys classical information to a receiver using a quantum-mechanical channel. We assume that each message corresponds to a pure quantum state $\{|\phi_i\rangle, 1 \leq i \leq m\}$. The information is detected by subjecting the system to a quantum measurement, optimized to distinguish between pure nonorthogonal states.

In unambiguous quantum detection [1] a measurement is designed comprising $m+1$ operators $\{\Pi_i, 0 \leq i \leq m\}$ satisfying $\sum_{i=0}^{m} \Pi_i = I$, that with a certain probability returns an inconclusive result, but such that if the measurement returns an answer, then the answer is correct with probability one. Each operator $\Pi_i, 1 \leq i \leq m$ corresponds to detection of the corresponding state $|\phi_i\rangle, 1 \leq i \leq m$, and Π_0 corresponds to an inconclusive result.

An unambiguous measurement exists if and only if the states are linearly independent [2]. In this case, without loss of generality [3], $\Pi_i = p_i|\tilde{\phi}_i\rangle\langle\tilde{\phi}_i| \stackrel{\triangle}{=} p_i Q_i, 1 \leq i \leq m$, for some $p_i \geq 0$, where $|\tilde{\phi}_i\rangle \in \mathcal{U}$ are the unique vectors in the space $\mathcal{U}$ spanned by the vectors $|\phi_i\rangle$ satisfying $\langle\tilde{\phi}_i|\phi_k\rangle = \delta_{ik}, 1 \leq i, k \leq m$. If the state $|\phi_i\rangle$ is prepared with prior probability η_i, then the total probability of correctly detecting the state is

$$P_D = \sum_{i=1}^{m} \eta_i \langle\phi_i|\Pi_i|\phi_i\rangle = \sum_{i=1}^{m} \eta_i p_i. \quad (1)$$

The problem then is to choose $p_i \geq 0$ to maximize P_D subject to $\sum_{i=1}^{m} p_i Q_i \leq I$.

II. Optimal Detection and the EPM

As we show in [3], the problem of (1) can be formulated as a convex *semidefinite programming problem (SDP)* [4]. By exploiting the many well known algorithms for solving SDPs, which are guaranteed to converge to the global optimum, the optimal measurement can be computed very efficiently in polynomial time. Furthermore, the SDP formulation can be used to derive necessary and sufficient conditions for optimality on p_i, as incorporated in the following theorem.

Theorem 1 *Let Λ denote the scalars $p_i \geq 0, 1 \leq i \leq m$ that satisfy $\sum_{i=1}^{m} p_i Q_i \leq I$, and let Γ denote the matrices $X \geq 0$ and scalars $z_i \geq 0, 1 \leq i \leq m$ such that $Tr(Q_i X) - z_i = \eta_i$. Then $p_i \in \Lambda$ maximizes P_D if and only if there exists $X, z_i \in \Gamma$ such that $X(I - \sum_{i=1}^{m} p_i Q_i) = 0$ and $z_i p_i = 0, 1 \leq i \leq m$.*

A simple suboptimal measurement for unambiguous discrimination is the *equal-probability measurement (EPM)* in which $p_i = p, 1 \leq i \leq m$. Using Theorem 1 we can derive conditions under which the EPM is optimal. In particular, we must have that $p = \sigma_m^2$ where σ_m is the smallest singular value of the matrix Φ of columns $|\phi_i\rangle$. In addition, we have the following theorem.

Theorem 2 *Let Φ have an SVD $\Phi = U\Sigma V^*$, let $|v_i\rangle$ denote the columns of V^* and $v_i(k)$ the kth component of $|v_i\rangle$, and let s be the multiplicity of the smallest singular value σ_m. Then,*

1. *If $s = 1$ then the EPM is optimal if and only if $|v_i(m)|^2 = \eta_i$ for $1 \leq i \leq m$;*
2. *If $s > 1$ then the EPM is optimal if there exists $b_i \geq 0, 1 \leq i \leq s$ such that $Ab = \eta$ where b and η are the vectors of components b_i and η_i respectively, and*

$$A = \begin{bmatrix} |v_1(m)|^2 & \cdots & |v_1(m-s+1)|^2 \\ |v_2(m)|^2 & \cdots & |v_2(m-s+1)|^2 \\ \vdots & & \vdots \\ |v_m(m)|^2 & \cdots & |v_m(m-s+1)|^2 \end{bmatrix}; \quad (2)$$

3. *For an arbitrary state set, if we choose the prior probabilities $\eta = Ab$ where $b_i \geq 0$ are arbitrary coefficients satisfying $\sum_{i=1}^{m} b_i = 1$, then the EPM is optimal;*
4. *If $\langle\phi_i|(\Phi\Phi^*)^{t/2-1}|\phi_i\rangle = \eta_i a_t, 1 \leq i \leq m, 1 \leq t \leq m$ for some constants a_t, then the EPM is optimal.*

Chefles and Barnett [5] showed that the EPM is optimal in the special case in which the state vectors form a cyclic set. Using Theorem 2 we can generalize this result to a large class of state sets. Specifically, for *geometrically uniform state sets* $\mathcal{S} = \{|\phi_i\rangle = U_i|\phi\rangle, U_i \in \mathcal{G}\}$ where $\mathcal{G}$ is a group of unitary matrices, the EPM is optimal. For *compound geometrically uniform (CGU) state sets* $\mathcal{S} = \{|\phi_{ik}\rangle = U_i|\phi_k\rangle, 1 \leq i \leq l, 1 \leq k \leq r\}$ where the matrices $\{U_i, 1 \leq i \leq l\}$ are unitary and form a group $\mathcal{G}$, and the vectors $\{|\phi_k\rangle, 1 \leq k \leq r\}$ are the generating vectors, the EPM is optimal if

$$\langle\phi_k|(\Phi\Phi^*)^{t/2-1}|\phi_k\rangle = a_t, \quad 1 \leq k \leq r, 1 \leq t \leq m, \quad (3)$$

for some constants a_t. In particular, if the generating vectors $\{|\phi_k\rangle = V_k|\phi\rangle, 1 \leq k \leq r\}$ are themselves GU, and $U_p V_t = V_t U_p e^{j\theta(p,t)}$ for all t and p and arbitrary $\theta(p,t)$, then the EPM is optimal.

References

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