Nonredundant and Redundant Sampling with Arbitrary Sampling and Reconstruction Spaces

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Abstract

This paper introduces a general framework for sampling and reconstruction procedures based on a consistency requirement, introduced by Unser and Aldroubi in [1]. The procedures we develop allow for almost arbitrary sampling and reconstruction spaces, as well as arbitrary input signals. We first derive a nonredundant sampling procedure. We then introduce the new concept of oblique dual frame vectors, based on which we develop a redundant sampling procedure that can be used to reduce the quantization error when quantizing the measurements prior to reconstruction.

1 Introduction

Many methods exist for representing a signal by a sequence of numbers, which can be interpreted as measurements of the signal we wish to represent. For example, the measurements may be the inner products of the signal with a set of vectors that span some subspace $\mathcal{V}$, which is referred to as the sampling space [1]. The problem then is to reconstruct the signal from these measurements, using a set of vectors that span a subspace $\mathcal{W}$, which we refer to as the reconstruction space. If the signal we wish to reconstruct does not lie in $\mathcal{W}$, then it can not be perfectly reconstructed using only reconstruction vectors that span $\mathcal{W}$. Therefore, if we allow for signals out of $\mathcal{W}$, then we must relax the requirement for perfect reconstruction.

In [1] the authors introduce the concept of consistent reconstruction, in which the reconstructed signal is in general not equal to the original signal, but nonetheless yields the same measurements. Based on this requirement, they derive a new sampling procedure for the special case where the signals lie in L_2, and where the sampling and reconstruction spaces are not necessarily equal but are both generated by integer translates of appropriately chosen functions.

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In this paper we extend the results of [1] in several ways. First, we expand their results to a broader framework that does not require the sampling and reconstruction spaces to be generated by integer translates, and does not require the signals to lie in L_2, but rather can be applied to arbitrary subspaces of an arbitrary Hilbert space. This framework leads to some new sampling theorems, as well as further insight into the results of [1]. Second, we develop *redundant* sampling procedures in which the measurements constitute an overcomplete representation of the signal. These measurements correspond to inner products of the signal with a frame for $\mathcal{V}$, and reconstruction is obtained using vectors which form a frame for $\mathcal{W}$. To obtain a consistent reconstruction of the signal in this case, we develop a generalization of the well known dual frame operator [2], which we refer to as an *oblique dual frame operator*. If the measurements are quantized prior to reconstruction, then as we show the average power of the reconstruction error using this redundant procedure can be reduced by as much as the redundancy factor of the frame in comparison with the nonredundant procedure.

2 Consistent Reconstruction

We denote vectors in an arbitrary Hilbert space $\mathcal{H}$ by lowercase letters, and the elements of a sequence $c \in l_2$ by $c[k]$. The inner product between vectors $x, y \in \mathcal{H}$ is denoted by $\langle x, y\rangle$. Suppose we are given measurements $c[k]$ of a signal f that lies in an arbitrary Hilbert space $\mathcal{H}$. The measurements are obtained by taking the inner product of the signal with a set of sampling vectors s_k that form a basis for some subspace $\mathcal{V}$ of $\mathcal{H}$, which is referred to as the sampling space, so that $c[k] = \langle s_k, f\rangle$. The basis vectors are chosen[1] so that the sequence c is in l_2. We construct an approximation $\hat{f}$ of f using a given set of reconstruction vectors w_k that span a subspace $\mathcal{W} \subseteq \mathcal{H}$, which we refer to as the reconstruction space.

[1] This condition is satisfied if the basis forms a Riesz basis [2]; therefore, all bases used in the paper are assumed to be Riesz bases.

Thus, $\hat{f}$ has the form $\hat{f} = \sum_k d[k]w_k$ for some coefficients $d[k]$ that are a linear transformation of the measurements $c[k]$. We do not require the sampling space $\mathcal{V}$ and the reconstruction space $\mathcal{W}$ to be equal.

If f is any signal in $\mathcal{W}$, then we will show that it can be perfectly reconstructed from the measurements $c[k]$ using reconstruction vectors w_k, as long as $\mathcal{W}$ and $\mathcal{V}^\perp$, the orthogonal complement of $\mathcal{V}$ in $\mathcal{H}$, are disjoint. However if f does not lie in $\mathcal{W}$, then it cannot be perfectly reconstructed using only vectors in $\mathcal{W}$. Since we are allowing the space of signals $\mathcal{H}$ to be larger than the reconstruction space $\mathcal{W}$, we must replace the requirement for perfect reconstruction with a less stringent requirement. The sampling procedures we develop are based on a consistency requirement, introduced by Unser and Aldroubi in [1]. The idea is to reconstruct an approximation $\hat{f}$ of f that has the property that if we measure it using the measurement vectors s_k, then the measurements will be equal to the measurements $c[k]$ of f. A signal $\hat{f}$ with this property is called a consistent reconstruction of f.

Our problem therefore is to obtain a consistent reconstruction $\hat{f}$ of any signal f in $\mathcal{H}$, given the measurements $c[k]$ and the reconstruction vectors w_k. We will show that such a reconstruction is always possible provided that $\mathcal{W}$ and $\mathcal{V}^\perp$ are disjoint. This then implies[2] that if f lies completely in $\mathcal{W}$, and $\mathcal{W}$ and $\mathcal{V}^\perp$ are disjoint, then f can be perfectly reconstructed from the measurements $c[k]$. Therefore, our results can also be used to generate new sampling theorems that yield perfect reconstruction. We will illustrate these ideas in the context of a concrete example in Section 4. In that example $\mathcal{H}$ is the space of length N discrete-time sequences $x[n]$, the reconstruction space $\mathcal{W}$ is the space of length $M = 2M' + 1 < N$ sequences, and the sampling space $\mathcal{V}$ is the space of "bandlimited" sequences in $\mathcal{H}$ so that $x[n] \in \mathcal{V}$ if and only if $X[k] = 0$ for $M' < k < N - M'$, where $X[k]$ is the N point DFT of $x[n]$. Using our framework we obtain a consistent "time-limited" reconstruction of any signal in $\mathcal{H}$, so that the lowpass DFT coefficients of the time-limited sequence and the original sequence are equal. Furthermore, we show that since any signal in $\mathcal{W}$ can be perfectly reconstructed from its samples in $\mathcal{V}$, a time-limited sequence can be reconstructed from a lowpass segment of its DFT transform.

To consider the consistency requirement in more detail, suppose $\hat{f} = \sum_k d[k]w_k$ where $d = Hc$ for some linear transformation H: $l_2 \to l_2$. The measurement sequence c can be expressed using set transformation notation[3] as $c = S^*f$, where S: $l_2 \to \mathcal{H}$ is the set transformation corresponding to the vectors s_k. The reconstructed signal is then given by

$$\hat{f} = \sum_k d[k]w_k = Wd = WHc = WHS^*f, \qquad (1)$$

where W is the set transformation corresponding to the vectors w_k. Since W and S are given, the problem reduces to finding a transformation H that will result in a consistent reconstruction $\hat{f}$, *i.e.,* such that

$$S^*\hat{f} = S^*WHS^*f = S^*f. \qquad (2)$$

In order for (2) to hold for all f, H must be invertible. Otherwise there exists a nonzero sequence of measurements $a = S^*f$ that lies in the null space of H, which from (1) results in $S^*\hat{f} = 0$, contradicting (2).

It is straightforward to show that (2) is satisfied with an invertible H if and only if WHS^* is a projection operator, denoted by G. It can further be shown that the range space $\mathcal{R}(G) = \mathcal{W}$ and the null space $\mathcal{N}(G) = \mathcal{V}^\perp$. This follows from the fact that the vectors s_k and w_k form bases for $\mathcal{V}$ and $\mathcal{W}$ respectively, and that H is invertible. Thus, G is a projection onto $\mathcal{W}$ along $\mathcal{V}^\perp$, denoted by $E_{\mathcal{W}\mathcal{V}^\perp}$. Since $\mathcal{W}$ is not necessarily equal to $\mathcal{V}$, this projection in general is not constrained to be an orthogonal projection, *i.e.,* it is an *oblique*[4] projection [3]. Thus, $\hat{f}$ is a consistent reconstruction of f only if

$$\hat{f} = E_{\mathcal{W}\mathcal{V}^\perp} f. \qquad (3)$$

If $\mathcal{W}$ and $\mathcal{V}^\perp$ are not disjoint, then $E_{\mathcal{W}\mathcal{V}^\perp}$ is not defined and consistent reconstruction for all $f \in \mathcal{H}$ is not possible.

3 Reconstruction From Nonredundant Measurements

We now show that if $\mathcal{W}$ and $\mathcal{V}^\perp$ are disjoint but otherwise arbitrary, then we can always obtain a consistent reconstruction of f from the measurements $c[k]$, *i.e.,* we can always find a transformation H such that $G = WHS^* = E_{\mathcal{W}\mathcal{V}^\perp}$. It can be shown that S^*W is invertible if and only if $\mathcal{W}$ and $\mathcal{V}^\perp$ are disjoint. In this case, reconstruction is obtained by first transforming the measurements $c[k]$ into "corrected" measurements $d[k]$ corresponding to the sequence $d = Hc = (S^*W)^{-1}c$. Then

$$\hat{f} = \sum_k d[k]w_k = Wd = W(S^*W)^{-1}S^*f. \qquad (4)$$

[2] The proof of this statement, as well as the proofs of many of the results presented in this paper, have been omitted in the interest of space.

[3] A *set transformation* X: $l_2 \to \mathcal{H}$ corresponding to the vectors $x_k \in \mathcal{H}$ is defined by $Xa = \sum_k a[k]x_k$ for any sequence $a \in l_2$. From the usual definition of the adjoint transformation X^*: $\mathcal{H} \to l_2$ it follows that if $a = X^*y$, then $a[k] = \langle x_k, y \rangle$.

[4] An oblique projection is a projection operator E satisfying $E^2 = E$ that is not necessarily Hermitian. The notation $E_{\mathcal{W}\mathcal{V}^\perp}$ denotes an oblique projection with range space $\mathcal{W}$ and null space $\mathcal{V}^\perp$. If $\mathcal{W} = \mathcal{V}$, then $E_{\mathcal{W}\mathcal{V}^\perp}$ is an orthogonal projection onto $\mathcal{W}$.

Comparing (4) with (3) we conclude that $\hat{f}$ of (4) is a consistent reconstruction of f only if $T = E_{\mathcal{W}\mathcal{V}^\perp}$, where $T = W(S^*W)^{-1}S^*$. From the definition of an oblique projection [3], $E_{\mathcal{W}\mathcal{V}^\perp}$ is the unique operator satisfying

$$\begin{aligned} E_{\mathcal{W}\mathcal{V}^\perp} w &= w \text{ for any } w \in \mathcal{W}; \\ E_{\mathcal{W}\mathcal{V}^\perp} v &= 0 \text{ for any } v \in \mathcal{V}^\perp. \end{aligned} \tag{5}$$

We can express any $w \in \mathcal{W}$ as $w = Wa$ for some $a \in l_2$, so that $Tw = TWa = Wa = w$. Since $s_k \in \mathcal{V}$, $S^*v = 0$ for any $v \in \mathcal{V}^\perp$, and $Tv = 0$. Thus T satisfies (5), and consequently $T = E_{\mathcal{W}\mathcal{V}^\perp}$, rendering $\hat{f} = Tf$ a consistent reconstruction of f. The resulting measurement and reconstruction scheme is depicted in Fig. 1.

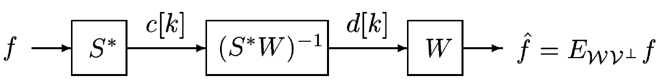

Figure 1: Consistent reconstruction of f using sampling vectors s_k and reconstruction vectors w_k, with $\mathcal{W}$ and $\mathcal{V}^\perp$ disjoint.

If $f \in \mathcal{W}$ then $\hat{f} = E_{\mathcal{W}\mathcal{V}^\perp} f = f$, and f can be perfectly reconstructed from the measurements $c[k]$ using the scheme depicted in Fig. 1. By choosing different spaces $\mathcal{H}$, $\mathcal{W}$ and $\mathcal{V}$ and using the measurement and reconstruction scheme of Fig. 1, we can arrive at a variety of new and interesting sampling theorems.

In [1] the authors consider the special case where $\mathcal{H} = L_2$ so that $f = f(t)$, and where $\mathcal{W}$ and $\mathcal{V}$ are subspaces generated by integer translates so that $w_k = \phi(t - kT)$ and $s_k = \psi(t - kT)$ for appropriately chosen functions $\phi(t), \psi(t)$. In this case the measurements $c[k] = \langle s_k, f(t) \rangle$ correspond to samples of the output of a filter with impulse response $\psi(-t)$ with $f(t)$ as its input, and the reconstructed signal corresponds to the output of a filter with impulse response $\phi(t)$, with an impulse train whose values are the corrected measurements $d[k]$ as its input. The transformation $(S^*W)^{-1}$ reduces to filtering with a filter whose impulse response is the convolutional inverse of the filter with impulse response $h_k = \langle s_k, w_0 \rangle = \langle \psi(t - kT), \phi(t) \rangle$. The sampling scheme of Fig. 1 then reduces to the sampling scheme of [1, Fig. 3]. However, our sampling scheme is more general than that of [1] since it applies to arbitrary subspaces $\mathcal{W}$ and $\mathcal{V}$ of an arbitrary Hilbert space $\mathcal{H}$. Furthermore, it is noted in [1] that consistent reconstruction is possible only if the filter with impulse response h_k is invertible. However, no general conditions are specified under which this is the case. From our general framework it follows that the filter will be invertible if and only if $\mathcal{W}$ and $\mathcal{V}^\perp$ are disjoint.

4 Bandlimited Sampling of Time-Limited Sequences

We now consider in detail the example outlined in Section 2. $\mathcal{H}$ is the space of sequences $x[n]$ such that $x[n] = 0$ for $n < 0, n \geq N$, $\mathcal{W}$ is the space of sequences $x[n]$ such that $x[n] = 0$ for $n < 0, n \geq M$ where $M = 2M' + 1 < N$, and $\mathcal{V}$ is the space of "bandlimited" sequences $x[n]$ such that $X[k] = 0$ for $M' < k < N - M'$, where $X[k], 0 \leq k \leq N-1$ denotes the N point DFT of $x[n]$. The bases for $\mathcal{V}$ and $\mathcal{W}$ are chosen as the sequences $s_k[n], 0 \leq k \leq M-1$ and $w_k[n], 0 \leq k \leq M-1$ respectively, given by $s_k[n] = e^{j2\pi(k-M')n/N}$ for $0 \leq n \leq N-1$ and 0 otherwise, and $w_k[n] = \delta[k-n]$.

Consider an arbitrary sequence $f[n]$ in $\mathcal{H}$. The measurements $c[k], 0 \leq k \leq M-1$ of $f[n]$ are equal to

$$\begin{aligned} c[k] = \langle s_k, f \rangle &= \sum_{n=0}^{N-1} s_k^*[n] f[n] \\ &= \sum_{n=0}^{N-1} f[n] e^{-j2\pi(k-M')n/N} = F[((k-M'))_N], \end{aligned} \tag{6}$$

where $F[k], 0 \leq k \leq N-1$ is the N point DFT of $f[n]$, and $((p))_N = p \bmod N$. Thus, the measurements $c[k]$ are the M lowpass DFT coefficients of the N point DFT of $f[n]$. To obtain a consistent reconstruction of $f[n]$ from $c[k]$ we need to determine $(S^*W)^{-1}$. The kmth element of the $M \times M$ matrix S^*W is

$$\langle s_k, w_m \rangle = \sum_{n=0}^{N-1} s_k^*[n] w_m[n] = s_k^*[m] = Z^{km} B^m, \tag{7}$$

where $Z = e^{-j2\pi/N}$ and $B = e^{j2\pi M'/N}$. We can therefore express S^*W in the form

$$S^*W = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & Z & Z^2 & \cdots & Z^{M-1} \\ 1 & Z^2 & Z^4 & \cdots & Z^{2(M-1)} \\ & & \vdots & & \\ 1 & Z^{M-1} & Z^{2(M-1)} & \cdots & Z^{(M-1)^2} \end{bmatrix} D. \tag{8}$$

Eq. (8) is the product of a Vandermonde matrix and a diagonal matrix D with nonzero diagonal elements $B^m, 0 \leq m \leq M-1$. Therefore, S^*W is always invertible from which we conclude that $\mathcal{W}$ and $\mathcal{V}^\perp$ are disjoint, so that consistent reconstruction is possible for all $f[n]$. We can compute the inverse of S^*W using any of the formulas for the inverse of a Vandermonde matrix (see *e.g.,* [4]). The corrected measurements $d[k]$ are then given by the elements of $d = (S^*W)^{-1}c$ where c is the vector with elements $c[k]$ given by (6), and $\hat{f}[n] = \sum_{k=0}^{N-1} w_k[n] d[k] = d_n$ for $0 \leq n \leq M-1$ and 0 otherwise. The consistency requirement implies that $\hat{F}[((k-M'))_N] = F[((k-M'))_N]$

for $0 \leq k \leq M-1$, where $\hat{F}[k]$ is the N point DFT of $\hat{f}[n]$. Thus $\hat{f}[n]$ is a "time-limited" sequence that has the same lowpass DFT coefficients as $f[n]$.

Now, suppose that $f[n]$ is a length M sequence in $\mathcal{W}$, and we are given M lowpass DFT coefficients $F[((k-M'))_N]$, $0 \leq k \leq M-1$. We can then perfectly reconstruct $f[n]$ from these coefficients using the method described above. This implies the intuitive result that a time-limited discrete-time sequence can be reconstructed from a lowpass segment of its DFT transform. This result is the analogue for the finite length discrete-time case of Papoulis' theorem [5], which implies that a time-limited function can be recovered from a lowpass segment of its Fourier transform. The reconstruction based on Papoulis' theorem is typically obtained using iterative algorithms such as that discussed in [5]. By choosing appropriate sampling and reconstruction vectors in the general scheme of Fig. 1, we obtained a finite length discrete-time version of this theorem together with a simple reconstruction method. This example illustrates the type of procedure that might be followed in using our framework to generate new sampling theorems.

5 Oblique Dual Frame Operator

In the previous sections we considered consistent reconstruction of a signal from a nonredundant set of measurements, given by inner products with a set of linearly independent vectors. We now consider the problem of consistent reconstruction from *redundant* measurements that are obtained using a linear dependent set of vectors that span the sampling space $\mathcal{V}$. The reconstruction is obtained using a set of linearly dependent vectors that span the reconstruction space $\mathcal{W}$. Such a set of vectors (with some additional constraints) is called a frame. More precisely, a set of vectors $y_k \in \mathcal{W}$ form a frame for $\mathcal{W}$ if there exists constants $A > 0$ and $B < \infty$ such that

$$A||x||^2 \leq \sum_k |\langle x, y_k \rangle|^2 \leq B||x||^2, \tag{9}$$

for all $x \in \mathcal{W}$ [2]. The lower bound in (9) ensures that the vectors y_k span $\mathcal{W}$. If the bounds $A = B$ in (9), then the frame is called a tight frame[5]. If $\mathcal{W}$ is an M-dimensional space and there are N vectors constituting the frame, then the redundancy of the frame is defined as $r = N/M$. Since the vectors y_k must span $\mathcal{W}$, $N \geq M$.

Suppose we are given a set of reconstruction vectors y_k that form a frame for $\mathcal{W}$. We can then reconstruct any signal $w \in \mathcal{W}$ using the frame vectors y_k, as $w = \sum_k \tilde{d}[k] y_k$ where the coefficients $\tilde{d}[k]$ are not uniquely specified. A possible choice is $\tilde{d}[k] = \langle \tilde{y}_k, w \rangle$ where $\tilde{y}_k$ are the *dual frame vectors* [2] of the frame vectors y_k, corresponding to the set transformation $\tilde{Y} = (Y^\dagger)^* = Y(Y^*Y)^\dagger$. Here Y is the set transformation corresponding to the vectors y_k, and $(\cdot)^\dagger$ denotes the *Moore-Penrose pseudo inverse* [6]. The operator $\tilde{Y}^* = Y^\dagger$ is called the *dual frame operator*. This choice of coefficients has the property that among all possible coefficients it has the minimal norm [2, 7]. If the vectors y_k form a tight frame, then $\tilde{Y} = Y$. From the properties of the pseudo inverse $YY^\dagger = P_{\mathcal{W}}$, where $P_{\mathcal{W}}$ is the orthogonal projection onto $\mathcal{W}$, so that indeed $\sum_k \langle \tilde{y}_k, w \rangle y_k = Y\tilde{Y}^* w = YY^\dagger w = P_{\mathcal{W}} w = w$. If f is any signal in $\mathcal{H}$, then $\hat{f} = \sum_k \langle \tilde{y}_k, f \rangle y_k = P_{\mathcal{W}} f$ is a consistent reconstruction of f from the redundant measurements $\tilde{d}[k] = \langle \tilde{y}_k, f \rangle$. Note, that here the sampling space is the space spanned by the vectors $\tilde{y}_k$ which is equal to the reconstruction space $\mathcal{W}$, spanned by the vectors y_k.

Now, suppose we want to reconstruct a vector $w \in \mathcal{W}$ using the reconstruction vectors y_k, but now the sampling space $\mathcal{V}$ is not necessarily equal to $\mathcal{W}$. In this case reconstruction can be obtained using what we define as the *oblique dual frame vectors* of y_k on $\mathcal{V}$, denoted by $\tilde{y}_k^{\mathcal{V}} \in \mathcal{V}$, which are the vectors in $\mathcal{V}$ corresponding to the set transformation $(Y^{\#}_{\mathcal{S}\mathcal{V}^\perp})^*$, where $\mathcal{S} = \mathcal{N}(Y)^\perp$ and $Y^{\#}_{\mathcal{S}\mathcal{V}^\perp}$ is the *oblique pseudo inverse* [8] of Y on $\mathcal{S}$ along $\mathcal{V}^\perp$. We define the *oblique dual frame operator* as the operator $Y^{\#}_{\mathcal{S}\mathcal{V}^\perp}$; when $\mathcal{V} = \mathcal{W}$, $Y^{\#}_{\mathcal{S}\mathcal{V}^\perp} = Y^\dagger$ and the oblique dual frame operator reduces to the conventional dual frame operator.

The oblique pseudo inverse is defined as follows. Let $T\colon \mathcal{K} \rightarrow \mathcal{U}$ be an arbitrary linear transformation, and let $\mathcal{K} = \mathcal{G} \oplus \mathcal{N}(T)$ and $\mathcal{U} = \mathcal{R}(T) \oplus \mathcal{Z}$. The *oblique pseudo inverse* of T on $\mathcal{G}$ along $\mathcal{Z}$, denoted $T^{\#}_{\mathcal{G}\mathcal{Z}}$, is the unique transformation satisfying [8] $T^{\#}_{\mathcal{G}\mathcal{Z}} T v = v$ for all $v \in \mathcal{G}$, and $T^{\#}_{\mathcal{G}\mathcal{Z}} w = 0$ for all $w \in \mathcal{Z}$. These conditions imply that

$$TT^{\#}_{\mathcal{G}\mathcal{Z}} = E_{\mathcal{R}(T)\mathcal{Z}}; \tag{10}$$

$$T^{\#}_{\mathcal{G}\mathcal{Z}} T = E_{\mathcal{G}\mathcal{N}(T)}. \tag{11}$$

Now, let y_k be a frame for $\mathcal{W}$ corresponding to the set transformation Y, and let $Y^{\#}_{\mathcal{S}\mathcal{V}^\perp}$ be the oblique pseudo inverse of Y on $\mathcal{S} = \mathcal{N}(Y)^\perp$ along $\mathcal{V}^\perp$. Then since $\mathcal{R}(Y) = \mathcal{W}$, it follows from (10) that $YY^{\#}_{\mathcal{S}\mathcal{V}^\perp} = E_{\mathcal{W}\mathcal{V}^\perp}$, so that any $w \in \mathcal{W}$ can be expressed as $w = E_{\mathcal{W}\mathcal{V}^\perp} w = \sum_k \langle \tilde{y}_k^{\mathcal{V}}, w \rangle y_k$. Since the vectors $\tilde{y}_k^{\mathcal{V}}$ lie in $\mathcal{V}$, the sampling space is now equal to $\mathcal{V}$ while the reconstruction space is equal to $\mathcal{W}$. If f is any signal in $\mathcal{H}$, then

$$E_{\mathcal{W}\mathcal{V}^\perp} f = YY^{\#}_{\mathcal{S}\mathcal{V}^\perp} f = \sum_k \langle \tilde{y}_k^{\mathcal{V}}, f \rangle y_k, \tag{12}$$

and $\hat{f} = \sum_k \langle \tilde{y}_k^{\mathcal{V}}, f \rangle y_k$ is a consistent reconstruction of f from a set of redundant measurements, where the sam-

[5] In this paper, we consider a tight frame to be a frame with bounds $A = B = 1$.

pling space is not necessarily equal to the reconstruction space.

We note that given the frame vectors y_k for $\mathcal{W}$, there are many ways of choosing coefficients $\tilde{d}[k]$ that correspond to measurements of a signal f using a frame for $\mathcal{V}$, and such that $E_{\mathcal{W}\mathcal{V}^\perp} f = \sum_k \tilde{d}[k] y_k$. The particular choice $\tilde{d}[k] = \langle \tilde{y}_k^{\mathcal{V}}, f \rangle$ given by the oblique dual frame vectors has the minimal norm among all possible coefficients. Another property of the oblique dual frame vectors, which is similar to a property of the conventional dual frame vectors (see, *e.g.,* [7, pp. 88-89]), is the following. Suppose we want to reconstruct a signal in $\mathcal{W}$ from some given coefficients $b[k]$. Among all possible reconstruction vectors we seek the vectors that result in a reconstructed signal whose measurements using sampling vectors t_k that span a space not necessarily equal to $\mathcal{W}$, are as close as possible to $b[k]$. Then the optimal reconstruction vectors are the oblique dual frame vectors of t_k on $\mathcal{W}$, corresponding to the set transformation $(T^{\#}_{\mathcal{N}(T)\mathcal{W}})^*$, where T is the set transformation corresponding to the vectors t_k.

6 Reconstruction From Redundant Measurements

Suppose now we are given a set of redundant measurements $\tilde{c}[k] = \langle x_k, f \rangle$ of a signal $f \in \mathcal{H}$, where the vectors x_k form an arbitrary frame for $\mathcal{V}$. We want to obtain a consistent reconstruction of f using a set of reconstruction vectors y_k that form a frame for $\mathcal{W}$. From (12) it follows that we can obtain a consistent reconstruction using the corrected measurements $\tilde{d}[k] = \langle \tilde{y}_k^{\mathcal{V}}, f \rangle$, where the vectors $\tilde{y}_k^{\mathcal{V}}$ are the oblique dual frame vectors of y_k on $\mathcal{V}$. To transform the given measurements $\tilde{c}[k]$ into the corrected measurements $\tilde{d}[k]$, we relate the vectors $\tilde{y}_k^{\mathcal{V}}$ to the given frame vectors x_k. Let X be the set transformation corresponding to the vectors x_k. Then the oblique dual frame operator $Y^{\#}_{\mathcal{S}\mathcal{V}^\perp}$ can be expressed in terms of X and Y as

$$Y^{\#}_{\mathcal{S}\mathcal{V}^\perp} = (X^*Y)^\dagger X^*. \quad (13)$$

Substituting (13) into (12) it follows that

$$\hat{f} = E_{\mathcal{W}\mathcal{V}^\perp} f = Y(X^*Y)^\dagger X^* f = Y(X^*Y)^\dagger \tilde{c} = Y\tilde{d}, \quad (14)$$

where $\tilde{d} = (X^*Y)^\dagger \tilde{c}$. Thus the corrected redundant measurements $\tilde{d}[k]$ are obtained by transforming the redundant measurements $\tilde{c}[k]$ using the transformation $(X^*Y)^\dagger$. The resulting sampling procedure is depicted in Fig. 2.

An alternative form of Fig. 2, that we will use later on, can be obtained by noting that any frame y_k for $\mathcal{W}$ can be expressed as $Y = WZ$ for some transformation Z: $l_2 \to l_2$, where W is a set transformation corresponding to an arbitrary basis for $\mathcal{W}$, and $ZZ^\dagger = I$. Similarly, $X = ST$ for some transformation T: $l_2 \to l_2$, where S is a set transformation corresponding to an arbitrary basis for $\mathcal{V}$, and $TT^\dagger = I$. Substituting into (13), we have

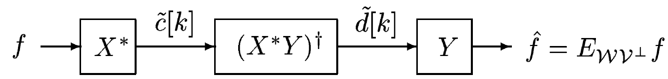

Figure 2: Consistent reconstruction of f using redundant sampling and reconstruction vectors x_k and y_k.

$$Y^{\#}_{\mathcal{S}\mathcal{V}^\perp} = Z^\dagger (S^*W)^{-1} (T^\dagger)^* T^* S^* = Z^\dagger (S^*W)^{-1} S^*, \quad (15)$$

since $(T^\dagger)^* T^* = I$. If in addition Y is a tight frame for $\mathcal{W}$, then $Z^\dagger = Z^*$ and

$$Y^{\#}_{\mathcal{S}\mathcal{V}^\perp} = Z^* (S^*W)^{-1} S^*. \quad (16)$$

From (15) it follows that we can obtain the redundant corrected measurements $\tilde{d}[k]$ directly from the nonredundant corrected measurements $d = (S^*W)^{-1} S^* f = (S^*W)^{-1} c$, via $\tilde{d} = Z^\dagger d$, where $c[k] = \langle s_k, f \rangle$ are the nonredundant measurements obtained using the vectors s_k. This interpretation is illustrated in Fig. 3.

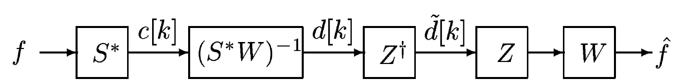

Figure 3: Equivalent form of Fig. 2.

One of the reasons for using redundant measurements is to reduce the average power of the quantization error, when quantizing the corrected measurements prior to reconstruction. If the sampling and reconstruction spaces are equal, then we have seen that we can consistently reconstruct a signal f using a set of vectors y_k that form a tight frame for $\mathcal{W}$, where the corrected measurements are $\tilde{d}[k] = \langle y_k, f \rangle$. Alternatively, we can use a nonredundant scheme where the reconstruction vectors w_k form an orthonormal basis for $\mathcal{W}$, and the corrected measurements are $d[k] = \langle w_k, f \rangle$. Suppose now we quantize the measurements $\tilde{d}[k]$ and $d[k]$ prior to reconstruction. Then using the redundant procedure, *i.e.,* quantizing the measurements $\tilde{d}[k]$ we can reduce the quantization error by the redundancy factor of the frame [2] in comparison with quantizing the measurements $d[k]$. We now extend this result to the case where the sampling and reconstruction spaces are not constrained to be equal. In particular, we show that we can choose a tight frame y_k for $\mathcal{W}$ such that when using the redundant sampling procedure of Figs. 2

and 3 we can reduce the average power of the reconstruction error by the redundancy factor in comparison with the nonredundant scheme of Fig. 1.

We now assume that $\mathcal{W}$ and $\mathcal{V}$ are M-dimensional spaces. Let $\{w_k, 1 \leq k \leq M\}$ denote a set of reconstruction vectors that form an orthonormal basis for $\mathcal{W}$, and let $\{s_k, 1 \leq k \leq M\}$ denote a set of sampling vectors that form a basis for $\mathcal{V}$. Let $c[k] = \langle s_k, f\rangle$ denote nonredundant measurements of a signal f. The consistent reconstruction $\hat{f}$ of f is obtained using the corrected measurements $d[k]$ corresponding to $d = (S^*W)^{-1}c$, which can be expressed as $d[k] = \langle v_k, f\rangle$, where $\{v_k, 1 \leq k \leq M\}$ are the vectors corresponding to the set transformation $V = S(W^*S)^{-1}$. Thus,

$$\hat{f} = \sum_{k=1}^{M} \langle v_k, f\rangle w_k = \sum_{k=1}^{M} q[k]\langle \overline{v}_k, f\rangle \overline{w}_k, \qquad (17)$$

where $q[k] = \sqrt{a[k]b[k]}$, $a[k] = \langle w_k, w_k\rangle = 1$, $b[k] = \langle v_k, v_k\rangle$, $\overline{w}_k = w_k/\sqrt{a[k]}$, and $\overline{v}_k = v_k/\sqrt{b[k]}$.

Assume we quantize the normalized measurements $\overline{d}[k] = \langle \overline{v}_k, f\rangle$ prior to reconstruction, and model the quantization error as an additive zero mean white noise source, so that the quantized measurements are given by $\overline{d}[k]' = \overline{d}[k] + e[k]$ where $E(e[k]e[j]) = \sigma^2\delta_{kj}$. The reconstruction error is then $\epsilon = \sum_{k=1}^{M} q[k]e[k]\overline{w}_k$ and

$$E(\langle \epsilon, \epsilon\rangle) = \sigma^2 \sum_{k=1}^{M} q^2[k] = \sigma^2 \sum_{k=1}^{M} b[k] \triangleq D. \qquad (18)$$

Suppose now we use a redundant procedure so that we reconstruct the signal using a tight frame $\{y_k, 1 \leq k \leq N\}$ for $\mathcal{W}$, with redundancy N/M. Then $Y = WZ$ for some Z: $\mathcal{C}^M \rightarrow \mathcal{C}^N$ such that $ZZ^* = I_M$. From (16) it follows that the sampling vectors leading to consistent reconstruction correspond to the set transformation $X = (Y^{\#}_{\mathcal{S}\mathcal{V}^{\perp}})^* = S(W^*S)^{-1}Z$, so that

$$\hat{f} = \sum_{k=1}^{N} \langle x_k, f\rangle y_k = \sum_{k=1}^{N} \tilde{q}[k]\langle \overline{x}_k, f\rangle \overline{y}_k, \qquad (19)$$

where $\tilde{q}[k] = \sqrt{\tilde{a}[k]\tilde{b}[k]}$, $\tilde{a}[k] = \langle y_k, y_k\rangle$, $\tilde{b}[k] = \langle x_k, x_k\rangle$, $\overline{y}_k = y_k/\sqrt{\tilde{a}[k]}$, and $\overline{x}_k = x_k/\sqrt{\tilde{b}[k]}$. If we quantize the normalized redundant measurements $\langle \overline{x}_k, f\rangle$, then the average power of the reconstruction error is $\tilde{D} = \sigma^2 \sum_{k=1}^{N} \tilde{q}^2[k]$. We now show that we can choose a tight frame y_k such that $\tilde{D} = M/ND$.

Let $Y = W\tilde{\mathcal{F}}$, where $\tilde{\mathcal{F}}$ is an $N \times M$ matrix whose rows are equal to the first M rows of the $N \times N$ DFT matrix with elements $1/\sqrt{N}e^{-j2\pi km/N}$. Then $X = (Y^{\#}_{\mathcal{S}\mathcal{V}^{\perp}})^* = V\tilde{\mathcal{F}}$. Indeed, since $\tilde{\mathcal{F}}\tilde{\mathcal{F}}^* = I_M$, $YX^* = WV^* = E_{\mathcal{W}\mathcal{V}^{\perp}}$. We now show that for this choice of sampling and reconstruction vectors $\tilde{D} = M/ND$. First we note that $Y^*Y = \tilde{\mathcal{F}}^*\tilde{\mathcal{F}} = M/NI_M$. Thus, $\tilde{a}[k] = M/N$ for all k and $\tilde{D} = \sigma^2 \sum_k \tilde{a}[k]\tilde{b}[k] = M/N\sigma^2 \sum_k \tilde{b}[k]$. Next,

$$\sum_{k=1}^{N} \tilde{b}[k] = \mathrm{Tr}(X^*X) = \mathrm{Tr}(V^*V) = \sum_{k=1}^{M} b[k], \qquad (20)$$

so that $\tilde{D} = M/N\sigma^2 \sum_k b[k] = M/ND$.

Therefore, to reduce the quantization error in the sampling and reconstruction scheme of Fig. 1, we propose the following. Instead of directly quantizing the measurements $d[k]$ in Fig. 1, we first take the N point DFT of the length M sequence of measurements $d[k]$, and then quantize the DFT coefficients. The reconstructed signal is then a linear combination of the reconstruction vectors w_k, where the coefficients are the first M values of the inverse DFT transform of the quantized DFT coefficients. The overall scheme is equivalent to that depicted in Fig. 3 where $Z^{\dagger}$ is now the N point DFT transform, and Z is the inverse DFT transform that returns only the first M values. The quantization takes place after the DFT transform.

We can show that we can obtain the same reduction in quantization error using frames for $\mathcal{W}$ and $\mathcal{V}$ corresponding to set transformations of the form $Y = WT$ and $X = VT$ respectively, where T is any cyclic unitary matrix or diagonal unitary matrix.

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