ROBUST LINEAR ESTIMATION WITH COVARIANCE UNCERTAINTIES

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ABSTRACT

We consider the problem of estimating a random vector $\mathbf{x}$, with covariance uncertainties, that is observed through a known linear transformation $\mathbf{H}$ and corrupted by additive noise. We first develop the linear estimator that minimizes the worst-case mean-squared error (MSE) across all possible covariance matrices. Although the minimax approach has enjoyed widespread use in the design of robust methods, we show that its performance is often unsatisfactory. We then develop a competitive minimax approach in which we seek the linear estimator that minimizes the worst-case *regret*, namely, the worst-case difference between the MSE attainable using a linear estimator, ignorant of the signal covariance, and the optimal MSE attained using a linear estimator that knows the signal covariance. We demonstrate, through an example, that the minimax regret approach can improve the performance over the minimax MSE approach.

1. INTRODUCTION

The theory of estimation in linear models has been studied extensively, following the classical works of Wiener [1] and Kolmogorov [2]. A fundamental linear estimation problem is that of estimating a stationary random signal filtered by a known linear time invariant (LTI) channel, in additive stationary noise. When the signal and noise spectral densities are completely specified, the estimation filter minimizing the mean-squared error (MSE) is the well-known Wiener filter.

If the spectral densities deviate from the ones assumed, then the performance of the Wiener filter matched to the assumed spectral densities can deteriorate considerably [3]. In such cases, it is desirable to design a robust filter whose performance is reasonably good across all spectral densities in the uncertainty region.

The most common approach for designing robust estimation filters is in the spirit of the minimax MSE approach, initiated by Huber [4], in which the estimation filter is chosen to minimize the worst-case MSE over an appropriately chosen class of spectral densities [5, 6, 3]. The minimax approach, in which the goal is to optimize the worst-case performance, is one of the major techniques for designing robust systems, and has been applied to many problems in detection and estimation [7, 8].

We consider a finite-dimensional analogue of the classical Wiener filtering problem, in which we seek to estimate a random vector $\mathbf{x}$ observed through a known linear transformation $\mathbf{H}$ and corrupted by additive noise $\mathbf{w}$. If the signal and noise covariance matrices are completely specified, then the linear minimum MSE (MMSE) estimator of $\mathbf{x}$ for this problem is well known [9].

In many practical applications the covariance of the noise can be estimated within high accuracy. The signal, on the other hand, will typically have a broader correlation function, so that estimating this correlation from the data with high accuracy often necessitates a larger sample size than is available. Therefore, in this paper, we develop methods for designing robust estimators in the case in which the covariance of the noise is known precisely, but the covariance of the desired signal $\mathbf{x}$ is not completely specified.

Following the popular minimax approach, in Section 3, we seek the linear estimator that minimizes the worst case MSE over all possible covariance matrices. The resulting estimator, referred to as the minimax MSE estimator, is shown to be an MMSE estimator matched to the worst possible choice of covariance matrix. Although the minimax approach has enjoyed widespread use in the design of robust methods, its performance is often unsatisfactory. The main limitation of this approach is that it tends to be overly conservative since it optimizes the performance for the worst possible choice of unknowns. As we show in the context of an example in Section 5, this can often lead to degraded performance.

To improve the performance of the minimax MSE estimator, in Section 4, we propose a new competitive approach to robust estimation in which we seek a linear estimator whose performance is as close as possible to that of the optimal estimator for all possible values of the covariance matrix. Specifically, we seek the estimator that minimizes the worst-case *regret*, which is the difference between the MSE of the estimator, ignorant of the signal covariance, and the smallest attainable MSE with a linear estimator that knows the signal covariance. By considering the *difference* between the MSE and the optimal MSE rather than the MSE directly, we can counterbalance the conservative character of the minimax approach, as is evident in the example we consider in Section 5.

The minimax regret concept has recently been used to develop a linear estimator for estimating a *deterministic* unknown vector $\mathbf{x}$, in a linear model [10]. Similar competitive approaches have been used in a variety of other contexts, for example, universal source coding [11], hypothesis testing [12], and prediction [13].

For analytical tractability, in our development we restrict attention to the class of linear estimators. As is well known [9], if $\mathbf{x}$ and $\mathbf{w}$ are jointly Gaussian vectors with known covariance matrices, then the estimator that minimizes the MSE, among all linear and nonlinear estimators, is the linear MMSE estimator. In [14], we show that this property does not hold when minimizing the worst-case regret with covariance uncertainties, even in the Gaussian case. Nevertheless, we demonstrate that in many cases we do not loose much by confining ourselves to linear estimators, which provides additional justification for the restriction to linear estimators in the context of minimax regret estimation.

2. PROBLEM FORMULATION

We denote vectors in $\mathbb{C}^m$ by boldface lowercase letters and matrices in $\mathbb{C}^{n\times m}$ by boldface uppercase letters. The matrix $\mathbf{I}$ denotes the identity matrix of the appropriate dimension, $(\cdot)^*$ denotes the Hermitian conjugate, and $\hat{(\cdot)}$ denotes an estimated vector.

Consider the problem of estimating the unknown vector parameter $\mathbf{x}$ in the linear model

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{w}, \tag{1}$$

where $\mathbf{H}$ is a known $n \times m$ matrix with rank m, $\mathbf{x}$ is a zero-mean, length-m random vector with covariance matrix $\mathbf{C}_x$ and $\mathbf{w}$ is a zero-mean, length-n random vector with known positive definite covariance $\mathbf{C}_w$, uncorrelated with $\mathbf{x}$. We assume that we only have partial information about the covariance $\mathbf{C}_x$.

We seek to estimate $\mathbf{x}$ using a linear estimator so that $\hat{\mathbf{x}} = \mathbf{G}\mathbf{y}$ for some $m \times n$ matrix $\mathbf{G}$. We would like to design an estimator $\hat{\mathbf{x}}$ of $\mathbf{x}$ to minimize the MSE, which is given by

$$E(\|\hat{\mathbf{x}}-\mathbf{x}\|^2) = \mathrm{Tr}\left(\mathbf{G}\mathbf{C}_w\mathbf{G}^*\right) + \mathrm{Tr}\left(\mathbf{C}_x(\mathbf{I}-\mathbf{G}\mathbf{H})^*(\mathbf{I}-\mathbf{G}\mathbf{H})\right). \tag{2}$$

If $\mathbf{C}_x$ is known and positive definite, then the linear estimator minimizing (2) is the *MMSE estimator* [9], which can be expressed as

$$\hat{\mathbf{x}} = (\mathbf{H}^*\mathbf{C}_w^{-1}\mathbf{H} + \mathbf{C}_x^{-1})^{-1}\mathbf{H}^*\mathbf{C}_w^{-1}\mathbf{y}. \tag{3}$$

If $\mathbf{C}_x$ is unknown, then we cannot implement (3). Instead, we may seek the estimator that minimizes the *worst-case* MSE over all possible choices of $\mathbf{C}_x$ consistent with our prior information. To reflect the uncertainty in our knowledge of $\mathbf{C}_x$, we assume that $\mathbf{C}_x$ and $\mathbf{H}^*\mathbf{C}_w^{-1}\mathbf{H}$ have the same eigenvector matrix, and that each of the nonnegative eigenvalues $\delta_i \geq 0, 1 \leq i \leq m$ of $\mathbf{C}_x$ satisfies

$$l_i \leq \delta_i \leq u_i, \quad 1 \leq i \leq m, \tag{4}$$

where $l_i \geq 0$ and u_i are known.

The assumption that $\mathbf{C}_x$ and $\mathbf{H}^*\mathbf{C}_w^{-1}\mathbf{H}$ have the same eigenvector matrix is made for analytical tractability. If $\mathbf{x}$ is a stationary random vector and $\mathbf{H}$ represents convolution of $\mathbf{x}$ with some filter, then both $\mathbf{C}_x$ and $\mathbf{H}$ will be Toeplitz matrices and are therefore approximately diagonalized by the Fourier transform matrix, so that $\mathbf{C}_x$ and $\mathbf{H}^*\mathbf{C}_w^{-1}\mathbf{H}$ approximately have the same eigenvectors.

The model (4) is reasonable when the covariance is estimated from the data. Specifically, denoting by $\zeta_i = (u_i + l_i)/2, \epsilon_i = (u_i - l_i)/2$ for $1 \leq i \leq m$, the conditions (4) can equivalently be expressed as $\delta_i = \zeta_i + e_i$ where $e_i^2 \leq \epsilon_i^2, 1 \leq i \leq m$, so that each of the eigenvalues of $\mathbf{C}_x$ lies in an interval of length $2\epsilon_i$ around some nominal value ζ_i, which we can think of as an estimate of the ith eigenvalue of $\mathbf{C}_x$ from the data vector $\mathbf{y}$. The interval specified by ϵ_i may be regarded as a confidence interval around our estimate ζ_i, and can be chosen to be proportional to the standard deviation of the estimate ζ_i.

Given $\{\zeta_i\}$, a straightforward approach to estimating $\mathbf{x}$ is to use an MMSE estimate corresponding to the estimated covariance. However, as we demonstrate through an example in Section 5, by taking an uncertainty interval around ζ_i into account, and seeking a competitive minimax estimator in this interval, we can further improve the estimation performance.

In Section 3, we develop the minimax estimator that minimizes the worst case MSE over all covariance matrices $\{\mathbf{C}_x\}$ that satisfy (4). The resulting estimator is an MMSE estimator matched to the worst possible choice of eigenvalues *i.e.*, $\delta_i = u_i$, and therefore tends to be overly conservative, which can often lead to degraded performance, as is evident in the example in Section 5. In this example, the minimax MSE estimator performs worse than the "plug-in" estimator, which is the MMSE estimator matched to the estimated covariance matrix.

To improve the performance of the minimax estimator, in Section 4 we consider a competitive approach in which we seek the linear estimator that minimizes the worst-case regret. The resulting estimator can also be interpreted as an MMSE estimator matched to a covariance matrix which depends on the nominal value ζ_i and the uncertainty interval ϵ_i, as well as on the eigenvalues of $\mathbf{H}^*\mathbf{C}_w^{-1}\mathbf{H}$. In the example in Section 5, we demonstrate that the minimax regret estimator can improve the performance over both the minimax MSE estimator and the plug-in MMSE estimator.

3. MINIMAX MSE ESTIMATOR

We first seek the linear estimator that minimizes the worst-case MSE over all possible values of $\mathbf{C}_x$ that have the same eigenvector matrix as $\mathbf{H}^*\mathbf{C}_w^{-1}\mathbf{H}$, and with eigenvalues δ_i satisfying (4). Thus, let $\mathbf{H}^*\mathbf{C}_w^{-1}\mathbf{H}$ have an eigendecomposition $\mathbf{H}^*\mathbf{C}_w^{-1}\mathbf{H} = \mathbf{V}\Lambda\mathbf{V}^*$, where $\mathbf{V}$ is a unitary matrix and Λ is a diagonal matrix with strictly positive diagonal elements $\{\lambda_i\}$. Then $\mathbf{C}_x$ has the form

$$\mathbf{C}_x = \mathbf{V}\Delta\mathbf{V}^*, \tag{5}$$

where Δ is a diagonal matrix with strictly positive diagonal elements $\{\delta_i\}$, with $l_i \leq \delta_i \leq u_i, 1 \leq i \leq m.$

We now consider the problem

$$\begin{aligned}\min_{\mathbf{G}} \max_{l_i \leq \delta_i \leq u_i} & E(\|\mathbf{G}\mathbf{y}-\mathbf{x}\|^2) = \\ & \min_{\mathbf{G}} \left\{\mathrm{Tr}(\mathbf{G}\mathbf{C}_w\mathbf{G}^*) + \max_{l_i \leq \delta_i \leq u_i} \mathcal{Q}(\mathbf{C}_x)\right\},\end{aligned} \tag{6}$$

where from (2),

$$\mathcal{Q}(\mathbf{C}_x) = \mathrm{Tr}\left(\mathbf{C}_x(\mathbf{I}-\mathbf{G}\mathbf{H})^*(\mathbf{I}-\mathbf{G}\mathbf{H})\right). \tag{7}$$

To find the covariance matrix $\mathbf{C}_x$ maximizing $\mathcal{Q}(\mathbf{C}_x)$, we rely on the following lemma [14].

Lemma 1. *Let* $\mathbf{W}, \mathbf{T}$ *and* $\mathbf{M}$ *be nonnegative definite matrices with* $\mathbf{W} \leq \mathbf{T}$. *Then Tr*$(\mathbf{M}\mathbf{W}) \leq$ *Tr*$(\mathbf{M}\mathbf{T})$.

If $\mathbf{C}_x$ is an arbitrary matrix of the form (5) with eigenvalues $l_i \leq \delta_i \leq u_i$, then

$$\mathbf{C}_x \leq \mathbf{V}\mathbf{Z}\mathbf{V}^*, \tag{8}$$

where $\mathbf{Z}$ is a diagonal matrix with diagonal elements u_i. This then implies, from Lemma 1, that

$$\begin{aligned}&\mathrm{Tr}\left(\mathbf{C}_x(\mathbf{I}-\mathbf{G}\mathbf{H})^*(\mathbf{I}-\mathbf{G}\mathbf{H})\right) \leq \\ &\quad \mathrm{Tr}\left(\mathbf{V}\mathbf{Z}\mathbf{V}^*(\mathbf{I}-\mathbf{G}\mathbf{H})^*(\mathbf{I}-\mathbf{G}\mathbf{H})\right),\end{aligned} \tag{9}$$

with equality if $\mathbf{C}_x = \mathbf{V}\mathbf{Z}\mathbf{V}^*$, so that $\mathcal{Q}(\mathbf{C}_x)$ is maximized for the worst possible choice of eigenvalues *i.e.*, $\delta_i = u_i$ for all i. The problem of (6), therefore, reduces to minimizing the MSE of (2) where we substitute $\mathbf{C}_x = \mathbf{V}\mathbf{Z}\mathbf{V}^*$. The optimal minimax MSE estimator is then the linear MMSE estimator of (3) with $\mathbf{C}_x = \mathbf{V}\mathbf{Z}\mathbf{V}^*$.

4. MINIMAX REGRET ESTIMATOR

To compensate for the conservative character of the minimax MSE approach, we now seek the linear estimator $\hat{\mathbf{x}}$ that minimizes the worst-case regret $\mathcal{R}(\mathbf{C}_x, \mathbf{G})$, which is defined as the difference between the MSE using an estimator $\hat{\mathbf{x}} = \mathbf{G}\mathbf{y}$ and the smallest possible MSE attainable with an estimator of the form $\hat{\mathbf{x}} = \mathbf{G}(\mathbf{C}_x)\mathbf{y}$ when the covariance $\mathbf{C}_x$ is known, denoted by MSE^o. If $\mathbf{C}_x$ is known, then the MMSE estimator is given by (3) and the resulting optimal MSE is

$$\mathrm{MSE}^o = \mathrm{Tr}\left((\mathbf{H}^*\mathbf{C}_w^{-1}\mathbf{H} + \mathbf{C}_x^{-1})^{-1}\right). \tag{10}$$

Thus, we seek the matrix $\mathbf{G}$ that is the solution to the problem

$$\min_{\mathbf{G}} \max_{l_i \le \delta_i \le u_i} \mathcal{R}(\mathbf{C}_x, \mathbf{G}), \tag{11}$$

where $\mathbf{C}_x$ has an eigendecomposition of the form (5), and

$$\begin{aligned}\mathcal{R}(\mathbf{C}_x, \mathbf{G}) &= \mathrm{Tr}(\mathbf{G}\mathbf{C}_w\mathbf{G}^*) + \mathrm{Tr}\left(\mathbf{C}_x(\mathbf{I} - \mathbf{G}\mathbf{H})^*(\mathbf{I} - \mathbf{G}\mathbf{H})\right) \\ &\quad - \mathrm{Tr}\left((\mathbf{H}^*\mathbf{C}_w^{-1}\mathbf{H} + \mathbf{C}_x^{-1})^{-1}\right). \end{aligned} \tag{12}$$

The linear estimator that minimizes the worst-case regret is given by the following theorem, the proof of which is derived in [14].

Theorem 1 (Minimax regret estimator). *Let $\mathbf{x}$ denote the unknown parameters in the model $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{w}$, where $\mathbf{H}$ is a known $n \times m$ matrix with rank m, $\mathbf{x}$ is a zero-mean random vector uncorrelated with $\mathbf{w}$ with covariance $\mathbf{C}_x$ and $\mathbf{w}$ is a zero-mean random vector with covariance $\mathbf{C}_w$. Let $\mathbf{H}^*\mathbf{C}_w^{-1}\mathbf{H} = \mathbf{V}\Lambda\mathbf{V}^*$ where $\mathbf{V}$ is a unitary matrix and Λ is an $m \times m$ diagonal matrix with diagonal elements $\lambda_i > 0$ and let $\mathbf{C}_x = \mathbf{V}\Delta\mathbf{V}^*$ where Δ is an $m \times m$ diagonal matrix with diagonal elements $0 \le l_i \le \delta_i \le u_i$. Then the solution to the problem*

$$\min_{\hat{\mathbf{x}} = \mathbf{G}\mathbf{y}} \max_{l_i \le \delta_i \le u_i} \left\{ E(\|\hat{\mathbf{x}} - \mathbf{x}\|^2) - \min_{\hat{\mathbf{x}} = \mathbf{G}(\mathbf{x})\mathbf{y}} E(\|\hat{\mathbf{x}} - \mathbf{x}\|^2) \right\}$$

is

$$\hat{\mathbf{x}} = \mathbf{V}\mathbf{C}\mathbf{V}^*\mathbf{H}^*\mathbf{C}_w^{-1}\mathbf{y},$$

where $\mathbf{C}$ is an $m \times m$ diagonal matrix with diagonal elements

$$c_i = \frac{1}{\lambda_i}\left(1 - \frac{1}{\sqrt{(1 + \lambda_i\zeta_i)^2 - \lambda_i^2\epsilon_i^2}}\right), \tag{13}$$

$\zeta_i = (u_i + l_i)/2$ and $\epsilon_i = (u_i - l_i)/2$.

As we now show, we can interpret the estimator of Theorem 1 as an MMSE estimator matched to a covariance matrix

$$\mathbf{C}_x = \mathbf{V}\mathbf{X}\mathbf{V}^*, \tag{14}$$

where $\mathbf{X}$ is a diagonal matrix with diagonal elements

$$x_i = \frac{1}{\lambda_i}\left(\sqrt{(1 + \lambda_i\zeta_i)^2 - \lambda_i^2\epsilon_i^2} - 1\right). \tag{15}$$

Note that if $\epsilon_i = 0$ so that the ith eigenvalue of the true covariance of $\mathbf{C}_x$ is equal to ζ_i then, as we expect, $x_i = \zeta_i$.

From (3), the MMSE estimate of $\mathbf{x}$ with covariance $\mathbf{C}_x$ given by (14) and $\mathbf{H}^*\mathbf{C}_w^{-1}\mathbf{H} = \mathbf{V}\Lambda\mathbf{V}^*$ is

$$\hat{\mathbf{x}} = \mathbf{V}\left(\Lambda + \mathbf{X}^{-1}\right)^{-1}\mathbf{V}^*\mathbf{H}^*\mathbf{C}_w^{-1}\mathbf{y}. \tag{16}$$

Since

$$\frac{1}{\lambda_i + \frac{1}{x_i}} = \frac{1}{\lambda_i}\left(1 - \frac{1}{\sqrt{(1 + \lambda_i\zeta_i)^2 - \lambda_i^2\epsilon_i^2}}\right) = c_i, \tag{17}$$

the estimator $\hat{\mathbf{x}}$ of (16) is equivalent to the estimator given by Theorem 1.

Since the minimax regret estimator minimizes the regret for $\mathbf{C}_x = \mathbf{V}\mathbf{X}\mathbf{V}^*$, we may view the covariance $\mathbf{C}_x = \mathbf{V}\mathbf{X}\mathbf{V}^*$ as the "least-favorable" covariance in the regret sense.

It is interesting to note that while the minimax MSE estimator for the model (4) is matched to a covariance matrix with eigenvalues $u_i \ge \zeta_i$, the minimax regret estimator is matched to a covariance matrix with eigenvalues $x_i \le \zeta_i$. Indeed, from (15),

$$x_i \le \frac{\sqrt{(\lambda_i\zeta_i + 1)^2} - 1}{\lambda_i} = \zeta_i. \tag{18}$$

Expressing x_i as

$$x_i = \frac{1}{\lambda_i}\left((1 + \lambda_i\zeta_i)\sqrt{1 - a_i} - 1\right), \tag{19}$$

where $a_i = \lambda_i^2\epsilon_i^2/(1 + \lambda_i\zeta_i)^2 < 1$, (since $\zeta_i \ge \epsilon_i$), and using the first order approximation $\sqrt{1-y} \approx 1 - (1/2)y$ for $0 \le y < 1$,

$$x_i \approx \zeta_i - \frac{\lambda_i\epsilon_i^2}{2(1 + \lambda_i\zeta_i)}. \tag{20}$$

Thus, the correction to the nominal covariance ζ_i is approximately $\lambda_i\epsilon_i^2/(2(1 + \lambda_i\zeta_i))$, which is quadratic in the length of the uncertainty interval ϵ_i.

5. EXAMPLE

We now consider an example illustrating the minimax regret estimator of Theorem 1. Suppose that

$$\mathbf{y} = \mathbf{x} + \mathbf{w}, \tag{21}$$

where $\mathbf{x}$ is a length-n segment of a zero-mean stationary first order AR process with components x_i, so that $E(x_i x_j) = \rho^{|j-i|}$ for some parameter ρ, and $\mathbf{w}$ is a zero-mean random vector uncorrelated with $\mathbf{x}$ with known covariance $\mathbf{C}_w = \sigma^2\mathbf{I}$. We assume that we know the model (21) and that $\mathbf{x}$ is a segment of a stationary process, however, its covariance $\mathbf{C}_x$ is unknown.

To estimate $\mathbf{x}$, we may first estimate $\mathbf{C}_x$ from the observations $\mathbf{y}$. A natural estimate of $\mathbf{C}_x$ is given by

$$\widehat{\mathbf{C}}_x = [\widehat{\mathbf{C}}_y - \mathbf{C}_w]_+ = [\widehat{\mathbf{C}}_y - \sigma^2\mathbf{I}]_+, \tag{22}$$

where

$$\widehat{\mathbf{C}}_y(i,j) = \frac{1}{n}\sum_{k=1}^{n-|j-i|} y_k y_{k+|j-i|} \tag{23}$$

is an estimate of the covariance of $\mathbf{y}$, and $[\mathbf{A}]_+$ denotes the matrix in which the negative eigenvalues of $\mathbf{A}$ are replaced by 0.

Given $\widehat{\mathbf{C}}_x$, we may estimate $\mathbf{x}$ using an MMSE estimate matched to $\widehat{\mathbf{C}}_x$, which we refer to as a plug-in estimator. However, as can be seen below in Fig. 1, we can further improve the estimation performance by using the minimax regret estimator.

To compute the minimax regret estimator, we choose $\mathbf{V}$ to be equal to the eigenvector matrix of the estimated covariance matrix

$\widehat{\mathbf{C}}_x$, and $\zeta_i = \sigma_i$ where σ_i are the eigenvalues of $\widehat{\mathbf{C}}_x$. We would then like to choose ϵ_i to reflect the uncertainty in our estimate ζ_i. Since computing the standard deviation of ζ_i is difficult, we choose ϵ_i to be proportional to the standard deviation of an estimator $\tilde{\sigma}_x^2$ of the variance σ_x^2 of $\mathbf{x}$, where

$$\tilde{\sigma}_x^2 = \frac{1}{n}\sum_{i=1}^{n} y_i^2 - \sigma_w^2. \quad (24)$$

Under the assumption that $\mathbf{x}$ and $\mathbf{w}$ are uncorrelated Gaussian random vectors, it can be shown that the variance of $\tilde{\sigma}_x^2$ is given by

$$E\left\{\left(\tilde{\sigma}_x^2 - \sigma_x^2\right)^2\right\} = \frac{2}{n}\left((\sigma_x^2+\sigma^2)^2 + \sum_{i=2}^{n} \mathbf{C}_x^2(1,i)\right). \quad (25)$$

Since σ_x^2 and $\mathbf{C}_x(1,i)$ are unknown, we substitute their estimates $\widehat{\mathbf{C}}_x(1,i), 1 \le i \le m$. Finally, to ensure that $\epsilon_i \le \zeta_i$, we choose

$$\epsilon_i = \min\left(\zeta_i, A\sqrt{\frac{2}{n}\left((\widehat{\mathbf{C}}_x^2(1,1)+\sigma^2)^2 + \sum_{i=2}^{n} \widehat{\mathbf{C}}_x^2(1,i)\right)}\right), \quad (26)$$

where A is a proportionality factor.

In Fig. 1, we plot the MSE of the minimax regret estimator averaged over 1000 noise realizations as a function of the SNR defined by $-10\log\sigma^2$ for $\rho = 0.8$, $n = 10$ and $A = 4$. The performance of the plug-in MMSE estimator matched to the estimated covariance matrix $\widehat{\mathbf{C}}_x$ and the minimax MSE estimator are plotted for comparison. As can be seen from the figure, the minimax regret estimator can increase the estimation performance particularly at low to intermediate SNR values. It is also interesting to note that the popular minimax MSE approach is useless in this example, since it leads to an estimator whose performance is worse than the performance of the plug-in estimator.

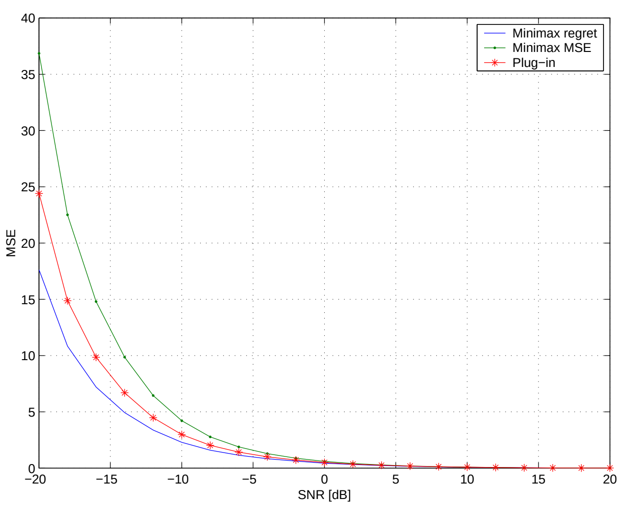

Fig. 1. MSE in estimating $\mathbf{x}$ as a function of SNR using the minimax regret estimator, the minimax MSE estimator and the plug-in MMSE estimator matched to the estimated covariance matrix.

6. CONCLUSION

We developed a competitive minimax approach for the problem of estimating a random vector $\mathbf{x}$ in the linear model $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{w}$, where the covariance matrix $\mathbf{C}_x$ of $\mathbf{x}$ is subject to uncertainties. In this approach, we seek the linear estimator that minimizes the worst-case regret, which is the difference between the MSE of the estimator and the best possible MSE attainable with a linear estimator that knows the covariance $\mathbf{C}_x$. As we demonstrated, the competitive minimax approach can increase the performance over the traditional minimax method, which in some cases turns out to be completely useless.

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