APPROXIMATING REPRESENTATION COEFFICIENTS FROM NON IDEAL SAMPLES

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ABSTRACT

Many sources of information are of analogue or continuous-time nature. However, digital signal processing applications rely on discrete data. We consider the problem of approximating L_2 inner products, *i.e.,* representation coefficients of a continuous-time signal, while having the possibly non-ideal signal samples, as the only available data. By adopting a generalized sampling scheme, a minimax solution is suggested. We then compare our approach with the piecewise-constant approximation method, commonly used for this problem.

1. INTRODUCTION

Signal processing applications are concerned mainly with digital data, although the origin of many sources of information is analogue. This is the situation in speech and audio, optics, radar, sonar, and biomedical signals.

In many cases, analysis of a continuous-time signal $\mathbf{x}(t)$ is obtained by evaluating L_2 inner-products $\langle \mathbf{w}_n(t), \mathbf{x}(t)\rangle$ for a set of functions $\{\mathbf{w}_n(t)\}$. As an example, suppose that a time-frequency analysis of a continuous-time signal is to be performed. To accomplish this, one may calculate a Gabor or wavelet representation. Both are based on finding the signal's representation coefficients by performing L_2 inner products with a known set of functions. In fact, due to the Riesz representation theorem [1], any linear and bounded functional applied to $\mathbf{x}(t)$ can be represented as inner product between some function $\mathbf{w}(t)$ and $\mathbf{x}(t)$.

In many applications of digital signal processing, however, there is no knowledge of the continuous-time signal itself, but only of its sample sequence. The problem is to approximate the required representation coefficients by proper processing of the available samples.

The case of processing uniform and ideal samples (*i.e.,* the n'th sample is $c[n] = \mathbf{x}(nT)$, where T is the sampling interval) was considered in [2]. In practice however, ideal sampling is impossible to implement. Here, we extend the work [2] by adopting the *generalized sampling* scheme [3, 4, 5, 6]. Generalized samples of a continuous-time signal are represented as the inner products of this signal with a set of sampling functions $\{\mathbf{s}_n(t)\}$ associated with the acquisition device. Thus, the n'th sample can be written as

$$c[n] = \langle \mathbf{s}_n(t), \mathbf{x}(t)\rangle. \tag{1}$$

This sampling model is general enough to describe any linear and bounded acquisition device. As an example, consider an analog to digital converter that performs pre-filtering prior to sampling. In such a setting, the sampling vectors $\{\mathbf{s}_n(t)\}$ are shifted and mirrored versions of the impulse response of the pre-filter [3].

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In many occasions, the fact that the samples are non-ideal is simply ignored. Assuming that the sample value is close to the mean value of the signal, within some interval of length T, a common approach is to approximate the L_2 inner product by a sum

$$\langle \mathbf{w}(t), \mathbf{x}(t)\rangle \approx T\sum_n c[n]\mathbf{w}^*(nT), \tag{2}$$

where * denotes complex conjugate. To determine the quality of this approximation, one must assume some regularity conditions on the functions involved. For analysis of this standard approach, see [7].

Yet another way to evaluate L_2 inner products, is to return first to the continuous-time domain [8, 9]. For that, the sample sequence is used as coefficients of some synthesis functions. In fact, this implies that the original signal is approximated within some subspace of L_2, created by the synthesis function set. For analysis of the latter methodology, see [9].

In this paper we consider a different approach. Given the generalized samples, we approximate the desired representation coefficients in a minimax sense. We then analyze the performance of the suggested method, comparing it with the sum approximation approach (2). Our results extend the work in [2] which treated the ideal sampling case, to practical non-ideal sampling schemes.

The outline of this paper is as follows. In Section 2 we describe the mathematical setting and formulate our problem. Section 3 develops the processing method of the available samples, by approximating the representation coefficients in a minimax sense. Section 4 evaluates the relations between the errors due to the minimax approach and the standard sum approximation method. Finally, in Section 5, we conclude with several simulations.

2. THE PROBLEM

2.1. Mathematical Preliminaries

We denote signals in L_2 by bold lowercase letters, omitting the time dependence, when possible. The elements of a sequence $c \in l_2$ will be written with square brackets, *e.g.* $c[n]$. The operator $P_{\mathcal{A}}$ represents the orthogonal projection onto a closed subspace $\mathcal{A}$ of L_2, and $\mathcal{A}^{\perp}$ is the orthogonal complement of $\mathcal{A}$. The *Moore-Penrose pseudo inverse* and the adjoint of a bounded transformation T are written as $T^{\dagger}$ and T^*, respectively. The inner product between signals $\mathbf{x}, \mathbf{y} \in L_2$ is $\langle \mathbf{x}, \mathbf{y}\rangle = \int_{-\infty}^{\infty} \mathbf{x}^*(t)\mathbf{y}(t)dt$, and $\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x}\rangle$ is the squared norm of $\mathbf{x}$. We denote by $\langle\cdot,\cdot\rangle_{l_2}$ and $\|\cdot\|_{l_2}$ the l_2 inner product and norm, respectively. S_T is the ideal sampling operator, such that the n'th element of $S_T\mathbf{w}$ is $\mathbf{w}(nT)$. A set transformation $V : l_2 \to L_2$ corresponding to frame vectors $\{\mathbf{v}_n(t)\} \in L_2$ is defined by $Va = \sum_n a[n]\mathbf{v}_n(t)$ for all $a \in l_2$. According to the definition of adjoint, if $a = V^*\mathbf{y}$, then $a[n] = \langle \mathbf{v}_n, \mathbf{y}\rangle$. Denoting

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by S (W) the set transformation corresponding to the vectors $\{\mathbf{s}_n\}$ ($\{\mathbf{w}_n\}$), the generalized samples $c[n] = \langle \mathbf{s}_n, \mathbf{x} \rangle$ can be written as $c = S^*\mathbf{x}$, and the desired representation coefficients $q[n] = \langle \mathbf{w}_n, \mathbf{x} \rangle$ by $q = W^*\mathbf{x}$.

To handle well posed problems, we assume that the generalized sample sequence c, and the required representation coefficients $q = W^*\mathbf{x}$ have finite energy, *i.e.,* $q, c \in l_2$. To ensure this, we assume that the set $\{\mathbf{s}_n\}$ ($\{\mathbf{w}_n\}$) forms a frame for $\mathcal{S} = \overline{span\,\{\mathbf{s}_n\}}$ ($\mathcal{W} = \overline{span\,\{\mathbf{w}_n\}}$).

2.2. Perfect Evaluation of Representation Coefficients

Our main goal is to approximate a set of representation coefficients $q = W^*\mathbf{x}$, by proper processing of the generalized samples $c = S^*\mathbf{x}$. A natural question to consider is whether there is an unavoidable error due to our partial knowledge of $\mathbf{x}(t)$, or can we evaluate **exactly** the required L_2 inner products based on the samples c. Let d be the sequence obtained by some (possibly nonlinear) transformation applied to the sample sequence. It is quite intuitive, and not hard to show that for a general $\mathbf{x} \in L_2$, the error can be made zero, *i.e.,* $q = d$, if and only if $\mathcal{W} \subseteq \mathcal{S}$. (See [10, Sec. 3] for proof of a similar result). However, by restricting our attention to a proper subgroup of signals in L_2, it is possible to obtain a zero error, even if $\mathcal{W} \nsubseteq \mathcal{S}$. This is true whenever there is a prior knowledge of $\mathbf{x}$, which enables us to determine a bijection between $\mathbf{x}(t)$ and its samples. To illustrate the last point, suppose that $\mathbf{x} \in \mathcal{A}$, where $\mathcal{A}$ is a closed subspace of L_2 satisfying the direct sum condition[1] $L_2 = \mathcal{A} \oplus \mathcal{S}^\perp$. Then, we can perfectly reconstruct $\mathbf{x}$ from its generalized samples, by [10]

$$\mathbf{x} = A(S^*A)^\dagger c, \tag{3}$$

where A is any bounded set transformation with the range $\mathcal{A}$. As a result, we can perfectly evaluate the coefficients $q = W^*x$ as

$$q = W^*A(S^*A)^\dagger c. \tag{4}$$

2.3. Problem Definition

In the general case, the condition $\mathcal{W} \subseteq \mathcal{S}$ may not be satisfied, or there is no prior knowledge of $\mathbf{x}(t)$. Thus, the coefficients W^*x cannot be computed exactly and instead must be approximated from the given samples c. A straightforward approach is to choose a sequence d that minimizes the squared norm of the error vector $W^*\mathbf{x} - d$. Since we know that $\mathbf{x}$ satisfies $c = S^*x$, by decomposing $\mathbf{x}$ along $\mathcal{S}$ and $\mathcal{S}^\perp$, the error vector can be written as $W^*S(S^*S)^\dagger c + W^*P_{\mathcal{S}^\perp}x - d$. This leads to the following objective

$$\min_d \left\| W^*S(S^*S)^\dagger c + W^*P_{\mathcal{S}^\perp}x - d \right\|_{l_2}^2. \tag{5}$$

However, it is not difficult to show that the solution of (5) will depend on $P_{\mathcal{S}^\perp}\mathbf{x}$, which is unknown. To cancel the dependence on $\mathbf{x}$, one may instead consider a robust approach, where the sequence d is optimized for the worst possible input $\mathbf{x}$. If the norm of the input is not bounded, so is the norm of the error. Hence, to define a well posed problem we will additionally assume that $\mathbf{x}$ is norm bounded by some positive constant L. This leads to the minimax objective

$$\min_d \max_{\|x\| \le L, c = S^*\mathbf{x}} \|W^*\mathbf{x} - d\|_{l_2}^2. \tag{6}$$

In the next sections we derive a solution for d, and compare its performance with the standard approach, given in (2).

[1] The direct sum between two closed subspaces $\mathcal{A}$ and $\mathcal{S}^\perp$ is the sum set $\{\mathbf{a} + \mathbf{v};\ \mathbf{a} \in \mathcal{A}, \mathbf{v} \in \mathcal{S}^\perp\}$ with the property $\mathcal{A} \cap \mathcal{S}^\perp = \{0\}$.

3. MINIMAX APPROXIMATION

The minimax problem of (6) is closely related to the generalized sampling problem considered in [10, Thr. 3].

Theorem 1. *Consider the problem*

$$\min_d \max_{c = S^*\mathbf{x}, \|x\| \le L} \|W^*\mathbf{x} - d\|_{l_2}^2,$$

where W and S are bounded set transformations with $\mathcal{R}(W) = \mathcal{W}$ and $\mathcal{R}(S) = \mathcal{S}$. A possible solution is

$$d = W^*S(S^*S)^\dagger c. \tag{7}$$

Proof. First we note that any $\mathbf{x}$ satisfying $S^*\mathbf{x} = c$ and $\|\mathbf{x}\| \le L$ is of the form $\mathbf{x} = S\,(S^*S)^\dagger\, c + \mathbf{v}$ for some $\mathbf{v} \in \mathcal{G}$ where

$$\mathcal{G} \triangleq \left\{ \mathbf{v} \mid \mathbf{v} \in \mathcal{S}^\perp, \|\mathbf{v}\| \le L' \right\},$$

and

$$L' = \sqrt{L^2 - \|S(S^*S)^\dagger c\|^2}. \tag{8}$$

Thus,

$$\begin{aligned} &\min_d \max_{c = S^*\mathbf{x}, \|x\| \le L} \|W^*\mathbf{x} - d\|_{l_2}^2 = \\ &\min_d \max_{\mathbf{v} \in \mathcal{G}} \left\| W^*S(S^*S)^\dagger c - d + W^*\mathbf{v} \right\|_{l_2}^2. \end{aligned} \tag{9}$$

The proof then follows from the proof of [10, Thr. 3], by redefining the vector a_d as it appears in [10, Thr. 3], to be $a_d = W^*S(S^*S)^\dagger c - d$, and using $W^*\mathbf{v}$ instead of $P_{\mathcal{W}}\mathbf{v}$. □

Note that (7) resembles the solution of the Wiener-Hopf equations, where the Gramian matrix of the autocorrelations is first inverted (pseudo-inverted), and the cross-correlation Gramian matrix is then applied. Also note, that we can rewrite (7) as $d = W^*P_{\mathcal{S}}\mathbf{x}$. Thus, the proposed approximation method results in a zero error if the prior knowledge of $\mathbf{x} \in \mathcal{S}$ is correct. In fact, by identifying A of (4) with S, the solutions indeed coincide.

The minimax objective (6) extends a recent work by Kirshner and Porat [2]. There, it was assumed that a single representation coefficient $\langle \mathbf{w}(t), \mathbf{x}(t) \rangle$ is to be approximated by processing an ideal sample sequence $S_T\mathbf{x}$. Reinterpreting [2], the approximation problem of [2, Thr. 3], can be restated using the following objective:

$$\min_b \max_{\|x\| \le L} |\langle \mathbf{w}, \mathbf{x} \rangle - \langle b, S_T\mathbf{x} \rangle_{l_2}|. \tag{10}$$

Note that the prior $c = S_T\mathbf{x}$ is not expressed in (10) and that the processing method is restricted to a linear form through $\langle b, S_T\mathbf{x} \rangle_{l_2}$. Yet, by degenerating our results to this ideal sampling case setting, it can be shown that (7) coincides[2] with the solution of (10), as it is expressed in [2, Thr. 3].

4. ERROR ANALYSIS

In this section we investigate the error due to the suggested minimax approach, comparing it with the sum approximation (2).

Let $e_{mx} = W^*\mathbf{x} - d$ be the error sequences due to the minimax approach, where d is given by (7). Rewriting $\mathbf{x} = S(S^*S)^\dagger c + P_{\mathcal{S}^\perp}\mathbf{x}$ and using (7), we have

$$e_{mx} = W^*P_{\mathcal{S}^\perp}\mathbf{x}. \tag{11}$$

[2] In fact, this is rather a delicate point as Sobolev space must be considered instead of the L_2 Hilbert space; however, the full discussion is beyond the scope of this paper.

Define e_{sum} to be the error sequence due to the sum method (2). The n'th element of e_{sum} satisfies

$$e_{sum}[n] = \langle \mathbf{w}_n, \mathbf{x} \rangle - T \langle S_T \mathbf{w}_n, c \rangle_{l_2}.$$

Accordingly, we can also write

$$e_{sum} = a + e_{mx}, \tag{12}$$

where

$$a[n] = \langle \mathbf{w}_n, S(S^*S)^\dagger c \rangle - T \langle S_T \mathbf{w}_n, c \rangle_{l_2} \tag{13}$$

expresses the difference between the two approximation methods. Note that since the sample sequence c is available, and so are T, S and W, the l_2 sequence a is known as well.

In the lemma below we obtain tight bounds for $\|e_{sum}\|_{l_2}^2 - \|e_{mx}\|_{l_2}^2$. Obviously, if this value is positive, then the minimax method is preferable to the sum approximation method, and vice versa.

Lemma 1. *The squared norm difference* $\|e_{sum}\|_{l_2}^2 - \|e_{mx}\|_{l_2}^2$ *lies within the tight bounds* $B_L \leq \|e_{sum}\|_{l_2}^2 - \|e_{mx}\|_{l_2}^2 \leq B_H$, *where*

$$\begin{aligned} B_L &= \|a\|_{l_2}^2 - 2\left|\langle a, e_{mx} \rangle_{l_2}\right| \\ B_H &= \|a\|_{l_2}^2 + 2\left|\langle a, e_{mx} \rangle_{l_2}\right|. \end{aligned} \tag{14}$$

Proof. Taking the squared norm of both sides of (12) and rearranging terms, we have

$$\|e_{sum}\|_{l_2}^2 - \|e_{mx}\|_{l_2}^2 = \|a\|_{l_2}^2 + 2\Re\left\{\langle a, e_{mx} \rangle_{l_2}\right\},$$

where $\Re\{\cdot\}$ denotes the real part. Additionally, we can bound

$$-\left|\langle a, e_{mx} \rangle_{l_2}\right| \leq \Re\left\{\langle a, e_{mx} \rangle_{l_2}\right\} \leq \left|\langle a, e_{mx} \rangle_{l_2}\right|.$$

We now show that those bounds are tight. Assume to the contrary that $e_{mx} = W^* P_{\mathcal{S}^\perp} \mathbf{x}$ maximizes $\Re\{\langle a, e_{mx} \rangle_{l_2}\}$, but

$$\Re\left\{\langle a, e_{mx} \rangle_{l_2}\right\} < \left|\langle a, e_{mx} \rangle_{l_2}\right|.$$

Define $\mathbf{x}_2 = S(S^*S)^\dagger c + \frac{\langle W^* P_{\mathcal{S}^\perp} \mathbf{x}, a \rangle_{l_2}}{\left|\langle W^* P_{\mathcal{S}^\perp} \mathbf{x}, a \rangle_{l_2}\right|} P_{\mathcal{S}^\perp} \mathbf{x}$. Note that $\mathbf{x}_2$ is a valid input since it satisfies the norm constraint $\|\mathbf{x}_2\| \leq L$ and is consistent with the known samples (*i.e.,* $c = S^* \mathbf{x}_2$). However, by examining the minimax error at $\mathbf{x}_2$, we have

$$\Re\left\{\langle a, W^* P_{\mathcal{S}^\perp} \mathbf{x}_2 \rangle_{l_2}\right\} = \left|\langle a, W^* P_{\mathcal{S}^\perp} \mathbf{x}_2 \rangle_{l_2}\right|,$$

contradicting our initial assumption. The proof of tightness for the lower bound is similar. □

Since the tight upper bound B_H is nonnegative for all choices of e_{mx}, we conclude that the sum approximation method cannot outperform the suggested minimax approach, for **all** possible inputs. Even when $T \to 0$, the minimax scheme can outperform the sum approximation approach by an amount not smaller than $\|a\|_{l_2}^2 = \|W^* P_{\mathcal{S}} \mathbf{x}\|_{l_2}^2$, as evident from (13) and (14).

On the other hand, in some cases, it is possible to have better performance by the minimax approach, for **all** possible inputs. To assure this, the lower bound B_L must be positive. In the following lemma, we introduce (without proof) a tight upper bound for $\|e_{mx}\|_{l_2}$ assuming that the set $\{\mathbf{w}_n(t)\}$ is orthonormal. Using this tight upper bound, we then present a sufficient condition for the minimax method to outperform the standard approach.

Lemma 2. *Let* $\{\mathbf{w}_n(t)\}$ *be an orthonormal set, and let* $\mathbf{x}$ *satisfy* $\|\mathbf{x}\| \leq L, c = S^* \mathbf{x}$. *Then*

$$\|e_{mx}\|_{l_2} \leq B_{mx} = \sqrt{1 - \cos^2(\mathcal{W}, \mathcal{S})} L',$$

where $\cos(\mathcal{W}, \mathcal{S}) = \inf_{\mathbf{y} \in \mathcal{W}, \|\mathbf{y}\|=1} \|P_{\mathcal{S}} \mathbf{y}\|$ *is related to the smallest angle [3, 10] between the spaces*[3] $\mathcal{W}, \mathcal{S}$ *and* L' *is given by (8).*

Using Lemma 1, Cauchy-Schwartz inequality, and Lemma 2 we state the following corollary.

Corollary 1. *Let* $\{\mathbf{w}_n(t)\}$ *be an orthonormal set. A sufficient condition for the minimax method to outperform the standard approach, for all possible inputs, is* $\|a\|_{l_2} \geq 2B_{mx}$.

Another interesting case, which is easy to evaluate, is when a single representation coefficient $\langle \mathbf{w}, \mathbf{x} \rangle_{L_2}$ is to be approximated. In such circumstances, e_{mx}, e_{sum} and a are all scalars. It can be shown that in such a setting, the minimax method and the sum approximation approach are tightly upper bounded by

$$|e_{mx}| \leq B_{mx} = L' \left\| P_{\mathcal{S}^\perp} \mathbf{w} \right\|, \tag{15}$$

$$|e_{sum}| \leq B_{sum} = |a| + B_{mx}. \tag{16}$$

A sufficient condition for the minimax method to outperform the sum approach becomes

$$|a| \geq 2L' \left\| P_{\mathcal{S}^\perp} \mathbf{w} \right\|. \tag{17}$$

As a conclusion from the above analysis, we get that when the spaces $\mathcal{W}$ and $\mathcal{S}$ are close, or when most of the signal's energy lies with the sampling space $\mathcal{S}$ (such that L' is small), then the minimax method will outperform the standard approach. Similarly, for large sampling intervals T, we can make $\|a\|_{l_2}$ large enough, again guaranteeing better performance by the minimax method.

5. EXAMPLES

Suppose that we wish to approximate a single representation coefficient $\langle \mathbf{w}, \mathbf{x} \rangle$, where $\mathbf{w}(t)$ is a normalized Gaussian. The generalized samples of $\mathbf{x}(t)$ were obtained by the average value of $\mathbf{x}(t)$ within a small interval of length Δ, *i.e.,*

$$c[n] = \frac{1}{\Delta} \int_{nT-\Delta}^{nT} \mathbf{x}(t) dt. \tag{18}$$

In this setting, the n'th sampling vector $\mathbf{s}_n(t)$ of (1) is

$$\mathbf{s}_n(t) = \begin{cases} 1/\Delta, & t \in [nT - \Delta, nT]; \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

The test signal $\mathbf{x}(t)$ is set to be a modulated and normalized Gaussian. Specifically, defining $\mathbf{y}(t) = \pi^{-\frac{1}{4}} e^{-t^2/2} \cos(4\pi t)$, we get $\mathbf{x}(t) = \mathbf{y}(t) / \|\mathbf{y}(t)\|$.

In the first experiment, we evaluated $B_{sum} - B_{mx}$ using the sample sequence c. As discussed in Section 4, this value describes the improvement of the minimax method over the standard approach, for the worst possible input $\mathbf{x}$. Note that the assumed test signal is by no means the worst possible input. It was merely used to produce the generalized samples. In Figure 1 we present the value of this bound difference for several choices of Δ. Note that the error is not monotone with respect to the sampling interval T. Additionally, when increasing the integration interval Δ the sum approximation

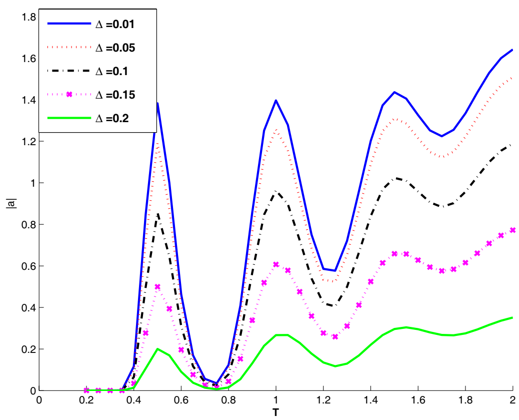

Fig. 1. Evaluation of $|a| = B_{sum} - B_{mx}$, which describes the possible improvement by the minimax method over the standard approach, for several choices of Δ.

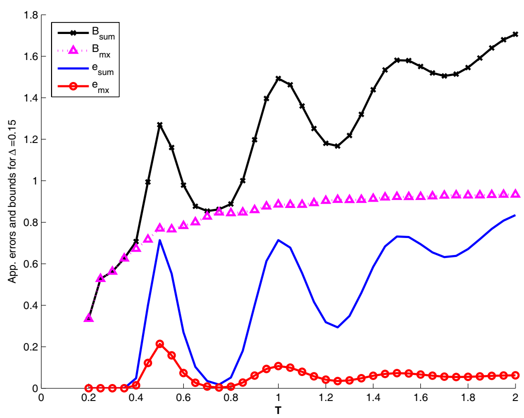

Fig. 2. Minimax vs. sum approximation. Shown are the upper bounds B_{mx}, B_{sum}, and the specific approximation errors e_{mx}, e_{sum} for $\Delta = 0.15$.

becomes closer to the true integral, which reduces the possible gain of the minimax method.

In the second experiment, shown in Figure 2, a comparison is made between the minimax method and the standard approach for the specific signals at hand. There, the integration interval Δ is set to 0.15. Surprisingly, the proposed technique outperforms the standard approach in terms of the concrete approximation error at hand. The latter happens despite the fact that condition (17) is not satisfied for this example. However, we remind the reader that (17) is a sufficient rather than a necessary condition.

6. CONCLUSIONS

A minimax approach has been introduced to approximating inner-product calculations within the continuous-time domain, when having only generalized samples of the signal as the available data. Comparison of the method with the standard Rieman sum approximation has been introduced. The derivations presented herein extend recent results concerning the ideal sampling case, allowing for practical sampling schemes to be considered.

[3]An explicit expression for $\cos(\mathcal{W}, \mathcal{S})$ in the case of *shift invariant* spaces, is given in [3].

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