ROBUST DECONVOLUTION OF NOISY SIGNALS

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ABSTRACT

We treat the problem of designing a robust estimation filter to recover a stationary random signal $x[n]$ convolved with a linear time-invariant (LTI) filter $h[n]$ and corrupted by additive stationary noise, in the presence of spectra uncertainties. Our approach is based on minimizing the worst-case difference between the MSE in the presence of uncertainties, and the MSE of the Wiener filter that knows the correct power spectra. The resulting filter, referred to as the *minimax regret filter*, takes the entire uncertainty interval into account, as well as the frequency response of the filter $h[n]$. We demonstrate through an example that the minimax regret filter can often lead to improved performance over traditional minimax MSE approaches for this problem.

1. INTRODUCTION

Deconvolution is aimed at removing the affects of a system on an input signal. A classical formulation of this problem is to recover a filtered, noisy signal assuming knowledge of the channel. This problem can be cast in the framework of estimation in a linear model in which the goal is to estimate the input signal $x[n]$ from corrupted observations $y[n]$ using a linear time invariant (LTI) estimation filter, where the signal is convolved with an LTI filter with impulse response $h[n]$, and corrupted by a stationary noise process $w[n]$.

If the power spectra of the signal and noise are known, then the deconvolution filter can be designed to minimize the mean-squared error (MSE), leading to the well-known Wiener filter [1]. However, if the power spectra are not completely specified, then the solution minimizing the MSE can not be obtained in general. An interesting problem that has attracted considerable attention in the literature is that of designing robust Wiener filters that have reasonable performance over all possible power spectra, in some region of uncertainty. The predominant approach is to choose the filter that minimizes the worst-case MSE over an appropriately chosen class of power spectra [2, 3, 4, 5, 6].

In Section 3, we consider the case where the signal and noise conform to a band uncertainty model defined by frequency-dependent known lower and upper bounds. As we show in Section 3.1, for this model, the standard minimax MSE filter is a Wiener filter matched to the upper bound on the power spectra, and is therefore overconservative. It also does not take the complete uncertainty region or the impulse response of the filter into account, since it depends only on the upper bound of the uncertainty region.

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Based on the estimation framework developed in [7, 8, 9] for the problem of estimating a finite-dimensional parameter vector from finitely many observations, we develop a competitive robust filter whose performance is uniformly close to that of the Wiener filter, for all possible values of the unknown power spectra. Specifically, in Section 3.2 we design a filter to minimize the worst-case *regret*, which is the difference between the MSE of the filter, ignorant of the signal and noise power spectra, and the smallest attainable MSE with a filter that knows the power spectra. By considering the *difference* between the MSE and the optimal MSE rather than the MSE directly, we can counterbalance the conservative character of the minimax MSE approach for this problem. In Section 4, we demonstrate through an example that the minimax regret approach can often lead to improved performance over traditional minimax MSE methods.

In the sequel, capital letters denote the discrete-time Fourier transform (DTFT), *e.g.,* $H(\omega)$ denotes the DTFT of $h[n]$, and $S_x(\omega)$ is the power spectrum of a random process $x[n]$. The complex conjugate is denoted by $(\cdot)^*$, and $\widehat{(\cdot)}$ denotes an estimated variable.

2. PROBLEM FORMULATION

We consider the basic deconvolution problem of recovering a zero-mean stationary random process $x[n]$ with power spectrum $S_x(\omega)$ from observations $y[n]$, where the sequence $y[n]$ is a filtered, noisy version of $x[n]$:

$$y[n] = h[n] * x[n] + w[n]. \tag{1}$$

Here $h[n]$ is a known filter with DTFT $H(\omega)$, and $w[n]$ is a wide-sense stationary noise process independent of $x[n]$, with zero-mean, and power spectrum $S_w(\omega)$. Our objective is to design a linear estimator $\hat{x}[n] = g[n] * y[n]$ of $x[n]$, where $g[n]$ is the impulse response of the estimation filter.

To construct an estimator $\hat{x}[n]$ that is close to $x[n]$, we may seek the filter $g[n]$ that minimizes the MSE $E\left\{|\hat{x}[n]-x[n]|^2\right\}$. For a given estimation filter $g[n]$ with DTFT $G(\omega)$, the MSE can be written as [1]

$$E(G,S_x,S_w) \stackrel{\triangle}{=} E\left\{|\hat{x}[n]-x[n]|^2\right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(|1-G(\omega)H(\omega)|^2 S_x(\omega) + |G(\omega)|^2 S_w(\omega)\right)d\omega. \tag{2}$$

If $S_x(\omega)$ and $S_w(\omega)$ are known, then the filter minimizing the MSE is the Wiener filter [1]:

$$G_{\mathrm{w}}(\omega) = \frac{H^*(\omega)S_x(\omega)}{S_w(\omega)+S_x(\omega)|H(\omega)|^2}. \tag{3}$$

The smallest attainable MSE, which is equal to the MSE of the Wiener filter, is

$$E(G_{\mathrm{w}}, S_x, S_w) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{S_w(\omega)S_x(\omega)}{S_w(\omega)+S_x(\omega)|H(\omega)|^2} d\omega. \quad (4)$$

In many practical applications $S_x(\omega)$ and $S_w(\omega)$ may not be known precisely, in which case the Wiener filter of (3) cannot be implemented. One possible approach in this case is to design a Wiener filter matched to the estimated power spectra. However, if the true power spectra deviate from the ones assumed, then the performance of the Wiener filter may deteriorate considerably [5]. Therefore, there is a need for a robust Wiener filter whose performance is reasonably good across all possible power spectra, in the region of uncertainty.

To reflect the uncertainty in our knowledge of $S_x(\omega)$ and $S_w(\omega)$ we assume that they belong to the set $\mathscr{D}$ defined by

$$\begin{aligned}\mathscr{D} = \{&S_x(\omega), S_w(\omega)| \\ &l(\omega) \leq S_x(\omega) \leq u(\omega), L(\omega) \leq S_w(\omega) \leq U(\omega)\}, \quad (5)\end{aligned}$$

where the bounds $l(\omega), u(\omega), L(\omega)$ and $U(\omega)$ are known, and $l(\omega), L(\omega) \geq 0$. For simplicity, we further assume that $l(\omega)|H(\omega)|^2 + L(\omega) > 0$.

The model $\mathscr{D}$ of (5) is reasonable when the power spectra are estimated from the data. Specifically, suppose we estimate the signal power spectrum as $S_x^0(\omega)$. We may then assume that the true power spectrum $S_x(\omega)$ lies in an uncertainty interval of length $2\varepsilon(\omega)$ around $S_x^0(\omega)$, where $\varepsilon(\omega) = (u(\omega) - l(\omega))/2$. The region specified by $\varepsilon(\omega)$ can be regarded as a confidence interval around $S_x^0(\omega)$ and may be chosen to be proportional to the standard deviation of $S_x^0(\omega)$.

Given an uncertainty set, the most common approach for developing robust Wiener filters is to seek a filter that minimizes the worst-case MSE in this region. However, as we show in Section 3.2, the minimax MSE approach tends to be overconservative and often does not lead to satisfactory performance. To improve its performance, we consider, in Section 3.2, a competitive approach, similar to that suggested in [7, 8] for a finite-dimensional analogue of the Wiener filtering problem. Instead of minimizing the worst-case MSE, we suggest minimizing the worst-case regret with respect to the optimal linear filter without uncertainty, where the regret is defined as the difference between the MSE of a filter ignorant of the true power spectra, and the optimal MSE attainable using a filter that knows the power spectra. The minimax regret filter is again a Wiener filter matched to a "least-favorable" pair of power spectra, which depend explicitly on the uncertainty interval and on the DTFT of the filter.

3. MINIMAX DECONVOLUTION

3.1 Minimax MSE Wiener Filter

We begin by choosing the estimator that minimizes the worst-case MSE over the set $\mathscr{D}$ defined by (5).

Since for all ω and for all power spectra in $\mathscr{D}$,

$$\begin{aligned}&|1-G(\omega)H(\omega)|^2 S_x(\omega) + |G(\omega)|^2 S_w(\omega) \\ &\quad \leq |1-G(\omega)H(\omega)|^2 u(\omega) + |G(\omega)|^2 U(\omega), \quad (6)\end{aligned}$$

we have that

$$\min_G \max_{S_x, S_w \in \mathscr{D}} E(G, S_x, S_w) = \min_G E(G, u, U), \quad (7)$$

where $E(G, S_x, S_w)$ is the MSE defined by (2). From (3), the minimax MSE filter is therefore

$$G(\omega) = \frac{H^*(\omega)u(\omega)}{u(\omega)|H(\omega)|^2 + U(\omega)}. \quad (8)$$

The filter (8) is overconservative, since it minimizes the MSE for the worst-possible choice of parameters. It also does not take the full uncertainty region into account, or the filter $h[n]$, but rather considers only the upper bound. To compensate for the conservative nature of the minimax MSE approach, and design a filter that takes all the given knowledge into account, we next develop a minimax regret filter.

3.2 Minimax Regret Wiener Filter

If the power spectra $S_x(\omega)$ and $S_w(\omega)$ are known, then the filter $G(S_x, S_w)$ minimizing the MSE is the Wiener filter of (3), and the smallest attainable MSE $E(G_{\mathrm{w}}, S_x, S_w)$ is given by (4). Note that the optimal MSE is a function of the power spectra $S_x(\omega)$ and $S_w(\omega)$. The regret $\mathscr{R}(S_x, S_w, G)$ is defined as the difference between the MSE using a filter $G(\omega)$ and the smallest possible MSE:

$$\mathscr{R}(S_x, S_w, G) = E(G, S_x, S_w) - E(G_{\mathrm{w}}, S_x, S_w). \quad (9)$$

To try and uniformly approach the optimal MSE in the presence of power spectra uncertainties, we seek an estimator that minimizes the worst-case regret:

$$\min_G \max_{S_x, S_w \in \mathscr{D}} \mathscr{R}(S_x, S_w, G), \quad (10)$$

where $\mathscr{D}$ is defined by (5). The minimax regret filter, that is the solution to (10), is given in the following theorem.

Theorem 1. *Let $x[n]$ be a zero-mean, stationary signal with power spectrum $S_x(\omega)$ in the model $y[n] = h[n] * x[n] + w[n]$, where $h[n]$ is a known filter with DTFT $H(\omega)$ and $w[n]$ is a zero-mean stationary noise process, independent of $x[n]$, with power spectrum $S_w(\omega)$. Let $\hat{x}[n] = g[n] * y[n]$ denote an estimate of $x[n]$ where $g[n]$ is a filter with DTFT $G(\omega)$, and let $\mathscr{D}$ denote the set of power spectra defined by (5). Then the minimax regret filter $G_{\mathrm{REG}}(\omega)$ that is the solution to*

$$\min_G \max_{S_x, S_w \in \mathscr{D}} \left\{ E\{|\hat{x}[n] - x[n]|^2\} - \min_{G(S_x, S_w)} E\{|\hat{x}[n] - x[n]|^2\} \right\}$$

is given by

$$\begin{aligned}G_{\mathrm{REG}}(\omega) &= \frac{H^*(\omega)}{\sqrt{U(\omega)+l(\omega)|H(\omega)|^2} + \sqrt{L(\omega)+u(\omega)|H(\omega)|^2}} \cdot \\ &\cdot \left(\frac{l(\omega)}{\sqrt{U(\omega)+l(\omega)|H(\omega)|^2}} + \frac{u(\omega)}{\sqrt{L(\omega)+u(\omega)|H(\omega)|^2}} \right).\end{aligned}$$

Before proving the theorem we note that if $L(\omega) = U(\omega)$ and $l(\omega) = u(\omega)$ so that $S_x(\omega)$ and $S_w(\omega)$ are known, then as we expect, $G_{\mathrm{REG}}(\omega)$ reduces to the Wiener filter of (3).

Proof. We develop the minimax regret filter by first expressing $G(\omega)$ as $|G(\omega)|e^{j\phi(\omega)}$, and noting that the regret $\mathscr{R}(S_x, S_w, G)$ depends on $\phi(\omega)$ only through the expression

$$\begin{aligned}|1-G(\omega)H(\omega)|^2 &= 1 + |G(\omega)H(\omega)|^2 \\ &= -2|G(\omega)H(\omega)|\cos(\phi(\omega) + \psi(\omega)), \quad (11)\end{aligned}$$

where $H(\omega) = |H(\omega)|e^{j\psi(\omega)}$. The minimum over $\phi(\omega)$ is achieved when $\phi(\omega) = -\psi(\omega)$, in which case

$$|1-G(\omega)H(\omega)|^2 = (1-|G(\omega)H(\omega)|)^2. \quad (12)$$

It remains to determine the optimal value of $|G(\omega)|$, which is the solution to

$$\min_{|G|} \max_{S_x,S_w \in \mathscr{D}} \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathscr{M}(|G|,S_x,S_w)d\omega, \quad (13)$$

where we defined

$$\mathscr{M}(|G|,S_x,S_w) = (1-|G(\omega)H(\omega)|)^2 S_x(\omega) + |G(\omega)|^2 S_w(\omega) - \frac{S_w(\omega)S_x(\omega)}{S_x(\omega)|H(\omega)|^2+S_w(\omega)}. \quad (14)$$

Since the constraint set $\mathscr{D}$ is separable in ω,

$$\min_{|G|} \max_{S_x,S_w \in \mathscr{D}} \left\{ \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathscr{M}(|G|,S_x,S_w)d\omega \right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\min_{|G|} \max_{S_x,S_w \in \mathscr{D}} \{\mathscr{M}(|G|,S_x,S_w)\} \right) d\omega. \quad (15)$$

For a fixed ω, let $g = |G(\omega)|, h = |H(\omega)|, \sigma_x = S_x(\omega)$ and $\sigma_w = S_w(\omega)$. Then, our problem becomes

$$\min_{g\geq 0} \max_{l\leq\sigma_x\leq u, L\leq\sigma_w\leq U} \left\{ (1-gh)^2\sigma_x + g^2\sigma_w - \frac{\sigma_w\sigma_x}{\sigma_x h^2+\sigma_w} \right\}, \quad (16)$$

where $h \geq 0$, $l = l(\omega), u = u(\omega), L = L(\omega)$ and $U = U(\omega)$. The solution to (16) can be obtained by similar arguments to those used in [8], leading to the following lemma.

Lemma 1. *The solution to the problem*

$$\min_{g\geq 0} \max_{l\leq\sigma_x\leq u, L\leq\sigma_w\leq U} \left\{ (1-gh)^2\sigma_x + g^2\sigma_w - \frac{\sigma_w\sigma_x}{\sigma_x h^2+\sigma_w} \right\} \quad (17)$$

is

$$g = \frac{h}{\sqrt{U+lh^2}+\sqrt{L+uh^2}} \left(\frac{l}{\sqrt{U+lh^2}} + \frac{u}{\sqrt{L+uh^2}} \right). \quad (18)$$

The proof of the theorem then follows from combining Lemma 1 with $\phi(\omega) = -\psi(\omega)$. □

3.3 Wiener Interpretation of the Regret Filter

By direct substitution it can be shown that the minimax regret filter can be interpreted as a Wiener filter matched to the power spectra

$$\begin{aligned} S_x(\omega) &= \alpha(\omega)l(\omega) + (1-\alpha(\omega))u(\omega); \\ S_w(\omega) &= \alpha(\omega)L(\omega) + (1-\alpha(\omega))U(\omega), \end{aligned} \quad (19)$$

where

$$\alpha(\omega) = \frac{\sqrt{L(\omega)+u(\omega)|H(\omega)|^2}}{\sqrt{L(\omega)+u(\omega)|H(\omega)|^2}+\sqrt{U(\omega)+l(\omega)|H(\omega)|^2}}. \quad (20)$$

Therefore, we can view the power spectra (19) as estimators of the true, unknown power spectra. Specifically, the signal spectrum $S_x(\omega)$ at a given frequency ω_0, is estimated as a weighted combination of the bounds $u(\omega_0)$ and $l(\omega_0)$, where the weights depend explicitly on the signal and noise uncertainty level at ω_0, and on the magnitude of the DTFT of the filter $|H(\omega_0)|$. The same holds true for the noise spectrum $S_w(\omega)$. Thus, in contrast with the minimax MSE filter, which is matched to power spectra that are equal to the upper bound, the minimax regret filter takes both the upper and lower bounds into account, as well as the DTFT $H(\omega)$. Since the minimax regret filter minimizes the regret for the power spectra given by (19), we may view these as the "least-favorable" power spectra in the regret sense.

Some insight into the least-favorable power spectra can be gained by considering the low and high SNR regions. If $l(\omega)|H(\omega)|^2 \gg U(\omega)$, then it can be shown that

$$S_x(\omega) \approx \sqrt{u(\omega)l(\omega)}, \quad (21)$$

which is the geometric average of the lower and upper bounds. If, on the other hand, $u(\omega)|H(\omega)|^2 \ll L(\omega)$, then

$$S_x(\omega) \approx \frac{l(\omega)\sqrt{L(\omega)}+u(\omega)\sqrt{U(\omega)}}{\sqrt{L(\omega)}+\sqrt{U(\omega)}}. \quad (22)$$

Similarly,

$$S_w(\omega) \approx \begin{cases} \frac{L(\omega)\sqrt{l(\omega)}+U(\omega)\sqrt{u(\omega)}}{\sqrt{l(\omega)}+\sqrt{u(\omega)}}, & U(\omega) \ll l(\omega)|H(\omega)|^2; \\ \sqrt{U(\omega)L(\omega)}, & L(\omega) \gg u(\omega)|H(\omega)|^2. \end{cases} \quad (23)$$

4. EXAMPLE

We now illustrate the performance of the minimax MSE and the minimax regret filters. Clearly, the behavior of these filters depends on the values of the unknown power spectra. If, for example, $S_x(\omega) = u(\omega)$ and $S_w(\omega) = U(\omega)$, then the minimax MSE filter will provide the best performance, since it minimizes the MSE for this choice of power spectra. As suggested in [8], one possible way of assessing the performance of the filters, is to compute the MSE at the output of each of the filters for the best possible choice of power spectra, the worst possible choice, and the nominal (average) choice. Obviously, the minimax MSE filter will optimize the performance for the worst choice. However, as we will see in the example below, the minimax regret filter often performs only slightly worse than the minimax MSE filter in the worst case, but can provide a substantial performance improvement for the best choice of power spectra.

Consider the estimation problem represented by the model (1), where $x[n]$ is a zero-mean stationary first order AR process with power spectrum

$$S_x^{o}(\omega) = \frac{1}{|1-\rho e^{j\omega}|^2} \quad (24)$$

for some parameter ρ, and $w[n]$ is a zero-mean, uncorrelated random process with variance σ^2, where we assume for simplicity that σ^2 is known. The filter $h[n]$ is an FIR filter:

$$h[0] = 1,\ h[\pm 1] = -7/16,\ h[n] = 0, |n| > 1. \quad (25)$$

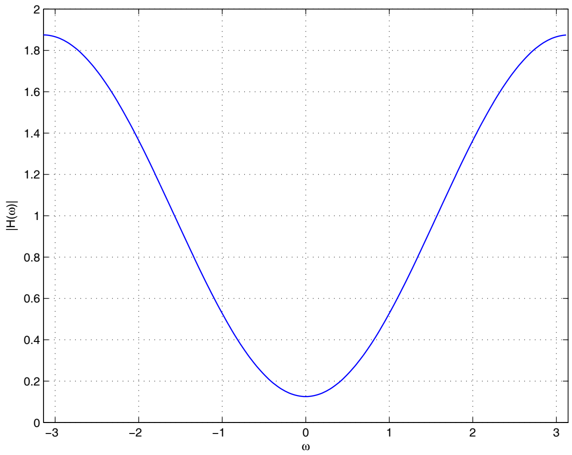

Figure 1: DTFT magnitude $|H(\omega)|$ of the filter given by (25).

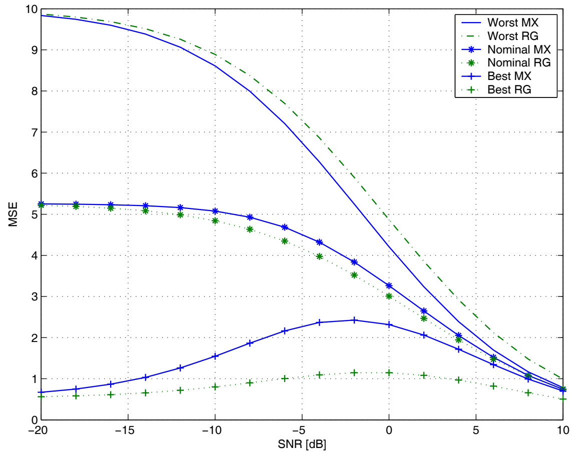

Figure 2: MSE as a function of SNR using the minimax regret (RG) and minimax MSE (MX) filters, for $S_x(\omega)$ equal to $S_x^{\text{WC}}(\omega)$ (Worst), $S_x^{\text{o}}(\omega)$ (Nominal), and $S_x^{\text{BC}}(\omega)$ (Best).

The DTFT magnitude of the filter is depicted in Fig. 1.

We assume that the signal spectrum $S_x(\omega)$ is not known exactly, however we know that $l(\omega) \leq S_x(\omega) \leq u(\omega)$ with $l(\omega) = (1-\alpha)S_x(\omega)$ and $u(\omega) = (1+\alpha)S_x(\omega)$, where $0 < \alpha < 1$ is a parameter that defines the size of the uncertainty.

For any estimation filter $G(\omega)$ we can find the worst choice of $S_x(\omega)$, denoted $S_x^{\text{WC}}(\omega)$, that maximizes the MSE, and the best choice of $S_x(\omega)$, denoted $S_x^{\text{BC}}(\omega)$, that minimizes the MSE. It is easy to see that the MSE defined by (2) is minimized when $S_x^{\text{BC}}(\omega) = l(\omega)$ and is maximized when $S_x^{\text{WC}}(\omega) = u(\omega)$, regardless of the filter $G(\omega)$.

In Fig. 2, we plot the MSE of the minimax MSE filter (MX) of (8) and the minimax regret filter (RG) of Theorem 1 as a function of the SNR defined by $-10\log\sigma^2$ for $\rho = 0.9$, and $\alpha = 0.9$. The MSE of each of the filters is plotted for three choices of $S_x(\omega)$: the worst case $S_x(\omega) = S_x^{\text{WC}}(\omega)$, the best case $S_x(\omega) = S_x^{\text{BC}}(\omega)$, and the nominal (true) value $S_x(\omega) = S_x^{\text{o}}(\omega)$. As we expect, when $S_x(\omega) = S_x^{\text{WC}}(\omega)$, the minimax MSE filter has the best performance. On the other hand, when $S_x(\omega) = S_x^{\text{BC}}(\omega)$, the performance of the minimax MSE filter deteriorates considerably. In this example, we may prefer using the minimax regret filter over the minimax MSE filter, since the loss in performance of the minimax MSE filter in the best case is much more significant then the loss in performance of the minimax regret filter in the worst case.

In Fig. 3 we plot the magnitude of the DTFTs of the minimax regret filter, the minimax MSE filter, and the inverse filter $G(\omega) = 1/H(\omega)$ for an SNR of 0dB.

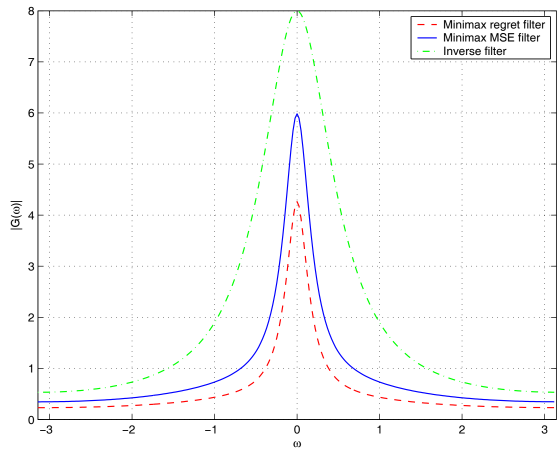

Figure 3: DTFT magnitude of the minimax regret, minimax MSE and inverse filters.

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