RECONSTRUCTION FROM FINITELY MANY SAMPLES IN THE PRESENCE OF NOISE

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ABSTRACT

We treat the class of sampling problems in which the underlying function can be specified by a finite set of samples. Our problem is to reconstruct the signal from non-ideal, noisy samples, which are modelled as the inner products of the signal with a set of sampling vectors, contaminated by noise. To mitigate the effect of the noise and the mismatch between the sampling and reconstruction vectors, the samples are linearly transformed prior to reconstruction. Considering a statistical reconstruction framework, we characterize the strategies that are mean-squared error (MSE) admissible, meaning that they are not dominated in terms of MSE by any other linear reconstruction. We also present explicit designs of admissible reconstructions that dominate a given inadmissible method. Adapting several estimation approaches to our problem, we suggest concrete admissible reconstruction methods and compare their performance. The results are then specialized to the case in which the samples are processed by a digital correction filter.

1. INTRODUCTION

Signal expansions, in which a signal is represented by a set of coefficients, find many applications in signal processing. Here, we focus on expansions with a finite number of coefficients. A natural setting in which expansions of this type arise is in the context of sampling a continuous-time signal that lies in a finite-dimensional space. For example, a band-limited periodic signal can be reconstructed from an arbitrary finite set of its samples, as long as the number of samples exceeds the corresponding Nyquist rate [1]. In practice, however, the signal samples are often contaminated by noise. The problem then is to "best" approximate the continuous-time signal from the given noisy samples in some sense.

In this paper, we consider the problem of recovering a signal x from a finite-set of its noisy samples. The signal is assumed to lie in an m-dimensional subspace $\mathcal{W}$ of an arbitrary Hilbert space $\mathcal{H}$. The noise-free samples $c_i = \langle s_i, x \rangle$ are described as inner products of x with a set of sampling vectors $\{s_i, 1 \leq i \leq m\}$. The reconstructed signal $\hat{x}$ is obtained by linearly combining a set of reconstruction vectors $\{w_i, 1 \leq i \leq m\}$ that span the signal space $\mathcal{W}$. The combination coefficients are the result of processing the noisy samples with a linear transformation designed to mitigate the effect of the noise and to compensate for the possible mismatch between the sampling and reconstruction vectors.

Previous methods for reconstruction in the presence of noise tend to focus on the bandlimited setting and are typically not specified to be optimal from the point of view of statistical estimation theory [2, 3, 4, 5]. Here, we study the reconstruction problem in a general Hilbert-space setting within a statistical framework.

A popular statistical measure of reconstruction performance is the mean-squared error (MSE), which is the average squared-norm of the estimation error $\hat{x} - x$. Unfortunately, the MSE performance of different reconstruction methods will generally depend on x, rendering comparison between different methods a difficult, and often impossible task. Nonetheless, a reconstruction $\hat{x}$ may *dominate* [6] some other approach $\hat{x}_0$ in terms of MSE, so that its MSE is never larger than that of $\hat{x}_0$ for all values of x, and is strictly smaller for some x; an *admissible* reconstruction is one that is never dominated by another reconstruction. Thus, although we cannot directly evaluate the MSE performance of different reconstruction strategies, we can characterize those methods which are admissible. Surprisingly, some of the previously proposed reconstruction approaches are inadmissible, implying that they can be uniformly improved upon in terms of MSE. We therefore suggest concrete designs of admissible reconstructions that dominate a given inadmissible method.

The paper is organized as follows. In Section 2 we introduce our general sampling framework, and show that both problems of admissibility and constructing dominating methods can be treated by solving a certain convex optimization problem. Dominating reconstructions and necessary and sufficient admissibility conditions on a bounded norm constraint set and the entire space are developed in Sections 3 and 5, respectively. In Section 4 we present several reconstruction methods on a bounded norm set by adapting known solutions to our particular sampling context: Least-squares, Tikhonov, and minimax MSE reconstruction [7]. Finally, in Section 6, we study FIR reconstruction and compare our results with the method of [3].

2. SAMPLING FRAMEWORK

We denote vectors in $\mathcal{H}$ by lowercase letters, and vectors in $\mathbb{C}^m$ by boldface lowercase letters. The ith element of a vector $\mathbf{a}$ is denoted by a_i. Matrices are represented by uppercase boldface letters and arbitrary linear transformations on $\mathcal{H}$ by uppercase letters. The orthogonal projection onto a space $\mathcal{S}$ is denoted by $P_{\mathcal{S}}$. Given a transformation T, $\mathcal{N}(T), \mathcal{R}(T)$ denote its null space and range space and $T^\dagger, T^*$ denote the pseudo inverse and the adjoint. For an operator A, $A \succ 0$ ($A \succeq 0$) means that A is Hermitian and positive (nonnegative) definite, and $A \succeq B$ means that $A - B \succeq 0$. The *set transformation* S: $\mathbb{C}^m \rightarrow \mathcal{H}$ corresponding to the vectors $\{s_i, 1 \leq i \leq m\}$ is defined by $S\mathbf{a} = \sum_{i=1}^m a_i s_i$ for any $\mathbf{a} \in \mathbb{C}^m$.

We treat the problem of reconstructing a signal $x \in \mathcal{W}$ from noisy samples $\mathbf{y} = S^*x + \mathbf{n}$, where $\mathbf{n}$ is a zero-mean noise vector with positive definite covariance matrix $\mathbf{C}$. The noise free samples

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$\mathbf{c} = S^*x$ are modelled as the inner products $c_i = \langle s_i, x\rangle$ of x with a set of linearly-independent sampling vectors $\{s_i, 1 \leq i \leq m\}$ with set transformation S, which span a subspace $\mathcal{S} \subseteq \mathcal{H}$. We consider reconstructions of the form

$$\hat{x} = \sum_{i=1}^{m} d_i w_i = W\mathbf{d} = W\mathbf{G}\mathbf{y}, \quad (1)$$

where $\mathbf{d} = \mathbf{G}\mathbf{y}$ is a linear transformation of the noisy samples $\mathbf{y}$, and W is the set transformation corresponding to a basis $\{w_i, 1 \leq i \leq m\}$ for $\mathcal{W}$. The sampling and reconstruction scheme is illustrated in Fig. 1.

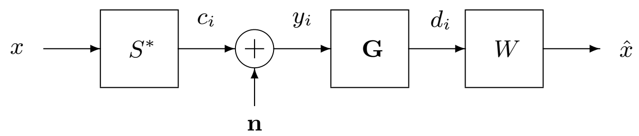

Fig. 1. General sampling and reconstruction scheme.

For the noise-free setting, it was shown in [8] that perfect reconstruction of any $x \in \mathcal{W}$ is possible as long as

$$\mathcal{W} \cap \mathcal{S}^{\perp} = \{0\}, \quad (2)$$

with

$$\mathbf{G} = \mathbf{G}_{\mathrm{CON}} = (S^*W)^{-1}. \quad (3)$$

The corresponding reconstruction $\hat{x}_{\mathrm{CON}} = W\mathbf{G}_{\mathrm{CON}}c = W(S^*W)^{-1}S^*x$, is referred to as a *consistent reconstruction*, since it yields the same samples as x for any $x \in \mathcal{H}$.

When the samples are corrupted by noise, $\mathbf{G} = \mathbf{G}_{\mathrm{CON}}$ no longer guarantees perfect reconstruction of $x \in \mathcal{W}$. Our problem then is to choose $\mathbf{G}$ such that $\hat{x}$ is close to x in some sense, for any $x \in \mathcal{W}$. Formulating the problem in a statistical framework, we seek the transformation $\mathbf{G}$ that minimizes the MSE $E\ \ \|\hat{x} - x\|^2$ between x and $\hat{x}$. However, computing the MSE shows that

$$E\ \ \|\hat{x} - x\|^2\ \ = \|(\mathbf{I} - W\mathbf{G}S^*)x\|^2 + \mathrm{Tr}(W^*W\mathbf{G}\mathbf{C}\mathbf{G}^*), \quad (4)$$

which depends in general on x and therefore cannot be minimized. To eliminate the signal dependency of the MSE we need to choose $\mathbf{G}$ such that $W\mathbf{G}S^*x = x$ for all $x \in \mathcal{W}$. The unique choice of $\mathbf{G}$ that satisfies this requirement is $\mathbf{G} = (S^*W)^{-1} = \mathbf{G}_{\mathrm{CON}}$; however, this method does not necessarily result in a small MSE.

Unless we use the consistent reconstruction strategy, the MSE of $\hat{x}$ in Fig. 1 will depend on x, and therefore cannot be evaluated. Nonetheless, some reconstructions may be better than others in an MSE sense. For example, we will see in Section 4 that if $\|x\| \leq L < \infty$, then based on a minimax MSE approach we can choose $\mathbf{G}$ such that the resulting MSE is smaller than the MSE of $\mathbf{G}_{\mathrm{CON}}$ *for all* $x \in \mathcal{W}, \|x\| \leq L$; thus, the minimax MSE reconstruction *dominates* the consistent reconstruction for norm bounded signals. A reconstruction $\hat{x}_1$ that dominates $\hat{x}_2$ is clearly preferable in terms of MSE. However, there may exists another reconstruction that dominates both $\hat{x}_1$ and $\hat{x}_2$. An *admissible* reconstruction has the property that it is not dominated by any other linear method. If a reconstruction is inadmissible, then there exists another approach which leads to better MSE performance on $\mathcal{U}$. This discussion raises two interesting questions:

1. Given a correction matrix $\mathbf{G}_0$, can we verify whether it leads to an admissible reconstruction on a set $\mathcal{U}$ of $\mathcal{W}$?
2. If $\mathbf{G}_0$ results in an inadmissible reconstruction, then can we develop a systematic approach for constructing an admissible method $\mathbf{G}$ that dominates $\mathbf{G}_0$ on $\mathcal{U}$?

Based on the results of [9], a general answer to both of these problems is given in the following theorem.

Theorem 1 *Let* $\mathbf{y} = S^*x + \mathbf{n}$ *denote noisy samples of a signal* x *in an* m*-dimensional subspace* $\mathcal{W} \subseteq \mathcal{H}$*, where* S *is a given set transformation and* $\mathbf{n}$ *is a zero-mean random vector with covariance* $\mathbf{C} \succ 0$*. Suppose that* $\hat{x}_0 = W\mathbf{G}_0\mathbf{y}$ *is a linear reconstruction of* x *in* $\mathcal{W}$*, where* W *is a given set transformation corresponding to a basis for* $\mathcal{W}$*. Denote the MSE of* $\hat{x}_0$ *by* $\epsilon(\mathbf{G}_0, x) = E\{\|W\mathbf{G}_0\mathbf{y} - x\|^2\}$*, and let* $\mathcal{U} \subseteq \mathcal{W}$*. Then*

1. $\hat{x}_0$ *is admissible on* $\mathcal{U}$ *iff* $\widehat{\mathbf{G}} = \mathbf{G}_0$ *where* $\widehat{\mathbf{G}}$ *is the unique solution to* $\min_{\mathbf{G}} \sup_{x\in\mathcal{U}} \{\epsilon(\mathbf{G}, x) - \epsilon(\mathbf{G}_0, x)\}$.
2. *If* $\min_{\mathbf{G}} \sup_{x\in\mathcal{U}} \{\epsilon(\mathbf{G}, x) - \epsilon(\mathbf{G}_0, x)\} < 0$ *then* $\hat{x} = W\widehat{\mathbf{G}}\mathbf{y}$ *strictly dominates* $\hat{x}_0$ *on* $\mathcal{U}$*;*
3. $\hat{x} = W\widehat{\mathbf{G}}\mathbf{y}$ *is admissible on* $\mathcal{U}$.

Theorem 1 provides a general recipe for determining admissibility of a linear reconstruction and for constructing admissible and strictly dominating reconstructions, by solving a convex optimization problem (the problem is convex in $\mathbf{G}$ for arbitrary sets $\mathcal{U}$ since the supremum of a convex function over any set $\mathcal{U}$ is convex.). In the rest of the paper we use Theorem 1 to develop easily verifiable admissibility conditions, as well as an explicit method for constructing a reconstruction dominating an inadmissible method.

3. NORM-BOUNDED SIGNALS

Suppose we are given a reconstruction $\hat{x}_0$ that is inadmissible on the set $\mathcal{U}$ of norm-bounded signals defined by

$$\mathcal{U} = \{x \in \mathcal{W}\, | \|x\| \leq L\}. \quad (5)$$

We would like to construct an admissible reconstruction $\hat{x}$ that strictly dominates $\hat{x}_0$. Applying Theorem 1, $\hat{x}$ can be constructed as $\hat{x} = W\widehat{\mathbf{G}}\mathbf{y}$ where

$$\widehat{\mathbf{G}} = \arg\min_{\mathbf{G}} \max_{x\in\mathcal{U}} \{\epsilon(\mathbf{G}, x) - \epsilon(\mathbf{G}_0, x)\}. \quad (6)$$

Optimality conditions on $\widehat{\mathbf{G}}$ are given in the following theorem.

Theorem 2 *The matrix* $\widehat{\mathbf{G}}$ *is the solution to (6) iff it satisfies the conditions below for some* $\lambda \geq 0$ *with* $R = W\widehat{\mathbf{G}}S^*P_{\mathcal{W}}$ *and* $R_0 = W\mathbf{G}_0S^*P_{\mathcal{W}}$*:*

1. $\mathbf{C}^{-1}S^*W\widehat{\mathbf{G}} = \widehat{\mathbf{G}}^*W^*S\mathbf{C}^{-1}$*;*
2. $0 \preceq \mathbf{C}^{-1}S^*W\widehat{\mathbf{G}} \prec \mathbf{C}^{-1}$*;*
3. $(P_{\mathcal{W}} - R)^*(P_{\mathcal{W}} - R) \preceq \lambda P_{\mathcal{W}} + (P_{\mathcal{W}} - R_0)^*(P_{\mathcal{W}} - R_0)$*;*
4. $(P_{\mathcal{W}} - R)^*(P_{\mathcal{W}} - R)R = \lambda R + (P_{\mathcal{W}} - R_0)^*(P_{\mathcal{W}} - R_0)R$*;*
5. $Tr(W^*W\widehat{\mathbf{G}}(\mathbf{C}^{-1} - \mathbf{C}^{-1}S^*W\widehat{\mathbf{G}})^{-1}(W^*S)^{-1}) \leq L^2$*;*
6. $\lambda Tr(W^*W\widehat{\mathbf{G}}(\mathbf{C}^{-1} - \mathbf{C}^{-1}S^*W\widehat{\mathbf{G}})^{-1}(W^*S)^{-1}) = \lambda L^2$.

Although the conditions of Theorem 2 are hard to solve in general, they can be used to verify a solution. For example, we can check when $\widehat{\mathbf{G}} = \mathbf{G}_0$, which leads to necessary and sufficient conditions for admissibility, as we discuss in Section 3.1. Furthermore, in some special cases the conditions can be solved explicitly, as in the case in which $P_{\mathcal{W}}S\mathbf{C}^{-1}S^*P_{\mathcal{W}}$ and $W\mathbf{G}_0S^*P_{\mathcal{W}}$ have the same eigenvector matrix.

Theorem 3 *Let $P_{\mathcal{W}} S \mathbf{C}^{-1} S^* P_{\mathcal{W}} = U \Sigma U^*$ where $\Sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_m)$ with $\sigma_i > 0$ and $U^* U = \mathbf{I}$, and suppose that $W \mathbf{G}_0 S^* P_{\mathcal{W}} = U \Delta U^*$ where $\Delta = \mathrm{diag}(\delta_1, \dots, \delta_m)$. Then*

$$\widehat{\mathbf{G}} = (W^* W)^{-1} W^* U \mathbf{D} U^* W (S^* W)^{-1}$$

where $\mathbf{D} = \mathrm{diag}(d_1, \dots, d_m)$ with $d_i = \max(1 - \sqrt{\eta_i}, 0)$ and $\eta_i = \lambda + |1 - \delta_i|^2$. The parameter λ is selected according to the following procedure: if $\mathcal{T}(0) \le 0$, where

$$\mathcal{T}(\lambda) = \sum_{i:\eta_i < 1} \frac{1}{\sigma_i} \quad \frac{1}{\sqrt{\eta_i}} - 1 \quad - L^2, \tag{7}$$

then $\lambda = 0$. Otherwise, λ is the unique value for which $\mathcal{T}(\lambda) = 0$ in the range $(0, \alpha)$ with $\alpha = 1 - \min_i |1 - \delta_i|^2$.

A general case in which the assumptions of the theorem are satisfied is when $\mathbf{C} = \sigma^2 \mathbf{I}$ for some $\sigma^2 > 0$, $P_{\mathcal{W}} S = W$, and $W^* W = I$ so that $\{w_i\}$ form an orthonormal basis for $\mathcal{W}$.

We now show that in the general case, the optimal $\widehat{\mathbf{G}}$ of (6) can be found numerically by solving an SDP [10], which is the problem of minimizing a linear objective subject to linear matrix inequality constraints. The advantage of this formulation is that it readily lends itself to efficient computational methods which are guaranteed to converge to the global optimum within any desired accuracy. In practice, the solution can be obtained using one of the many available SDP software packages.

To develop the SDP formulation we define the matrices $\mathbf{M} = U^* W \mathbf{G}$, $\mathbf{H} = S^* U$ and

$$\mathbf{A} = U^* (I - W \mathbf{G}_0 S^*)^* (I - W \mathbf{G}_0 S^*) U, \tag{8}$$

where $U\colon \mathbb{C}^m \to \mathcal{H}$ is the orthonormal set transformation in the singular value decomposition of W. The problem (6) can be formulated in terms of these matrices as

$$\min_{\mathbf{M}} \{\mathrm{Tr}(\mathbf{M}\mathbf{C}\mathbf{M}^*) + \max_{\|\mathbf{d}\| \le L} \mathbf{d}^* \left((\mathbf{I} - \mathbf{M}\mathbf{H})^* (\mathbf{I} - \mathbf{M}\mathbf{H}) - \mathbf{A} \right) \mathbf{d} \quad . \tag{9}$$

Now, for any matrix $\mathbf{Z}$, we have that

$$\max_{\|\mathbf{d}\| \le L} \mathbf{d}^* \mathbf{Z} \mathbf{d} \quad = \quad L^2 \max\left(\lambda_{\max}(\mathbf{Z}), 0\right), \tag{10}$$

where $\lambda_{\max}(\mathbf{Z})$ is the largest eigenvalue of $\mathbf{Z}$. Expressing $\max\left(\lambda_{\max}(\mathbf{Z}), 0\right)$ as the solution to

$$\min_{\lambda \ge 0} \{\lambda : \mathbf{Z} \preceq \lambda \mathbf{I}\}, \tag{11}$$

we can write (9) as

$$\min_{\tau, \mathbf{M}, \lambda \ge 0} \tau \tag{12}$$

subject to

$$\begin{aligned} \mathbf{m}^* \mathbf{m} + L^2 \lambda &\le \tau \\ (\mathbf{I} - \mathbf{M}\mathbf{H})^* (\mathbf{I} - \mathbf{M}\mathbf{H}) - \mathbf{A} &\preceq \lambda \mathbf{I}. \end{aligned} \tag{13}$$

Here $\mathbf{m} = \mathrm{vec}(\mathbf{C}^{1/2} \mathbf{M})$ is the vector obtained by stacking the columns of $\mathbf{C}^{1/2} \mathbf{M}$. Using Schur's complement [11, p. 472], the constraints (13) can be written as the linear matrix inequalities

$$\begin{matrix} \tau - L^2 \lambda & \mathbf{m}^* \\ \mathbf{m} & \mathbf{I} \end{matrix} \succeq 0$$

$$\begin{matrix} \lambda \mathbf{I} + \mathbf{A} & (\mathbf{I} - \mathbf{M}\mathbf{H})^* \\ \mathbf{I} - \mathbf{M}\mathbf{H} & \mathbf{I} \end{matrix} \succeq 0. \tag{14}$$

We conclude that the problem (9) is equivalent to the SDP of minimizing τ subject to (14). The solution $\widehat{\mathbf{G}}$ of (6) can be obtained from the SDP solution $\widehat{\mathbf{M}}$ as $\widehat{\mathbf{G}} = (U^* W)^{-1} \widehat{\mathbf{M}}$.

3.1. Admissible Reconstructions

We now use Theorem 2 to develop conditions on $\hat{x}_0$ to be admissible, or equivalently, such that $\mathbf{G}_0$ is the solution to (6):

Theorem 4 *The reconstruction $\hat{x} = W \widehat{\mathbf{G}} \mathbf{y}$ is admissible on $\mathcal{U}$ iff*

1. $\mathbf{C}^{-1} S^* W \widehat{\mathbf{G}} = \widehat{\mathbf{G}}^* W^* S \mathbf{C}^{-1}$;
2. $0 \preceq \mathbf{C}^{-1} S^* W \widehat{\mathbf{G}} \prec \mathbf{C}^{-1}$;
3. $\mathrm{Tr}(W^* W \widehat{\mathbf{G}} (\mathbf{C}^{-1} - \mathbf{C}^{-1} S^* W \widehat{\mathbf{G}})^{-1} (W^* S)^{-1}) \le L^2$.

An immediate consequence of Theorem 4 is that the consistent reconstruction $\hat{x}_{\mathrm{CON}} = W \mathbf{G}_{\mathrm{CON}} \mathbf{y}$ with $\mathbf{G}_{\mathrm{CON}}$ given by (3) is inadmissible on the bounded-norm set $\mathcal{U}$.

A general class of admissible reconstructions is given in the following proposition.

Proposition 1 *Let $P_{\mathcal{W}} S \mathbf{C}^{-1} S^* P_{\mathcal{W}} = U \Sigma U^*$ where $\Sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_m)$ with $\sigma_i > 0$. Then $\hat{x} = W \widehat{\mathbf{G}} \mathbf{y}$ is admissible for any $\widehat{\mathbf{G}}$ of the form*

$$\widehat{\mathbf{G}} = (W^* W)^{-1} W^* U \mathbf{D} U^* W (S^* W)^{-1}, \tag{15}$$

where $\mathbf{D} = \mathrm{diag}(d_1, \dots, d_m)$ with values d_i satisfying

$$0 \le d_i < 1, \quad 1 \le i \le m; \quad \sum_{i=1}^{m} \frac{d_i}{(1 - d_i)\sigma_i} \le L^2.$$

4. RECONSTRUCTION METHODS ON A SPHERE

From Theorem 4 we see that there are many possible admissible reconstructions on $\mathcal{U}$. Therefore, there is a need for a criterion to choose between them. In this section we consider several possible strategies and develop the corresponding solutions.

Least-Squares: The most straightforward approach to designing $\mathbf{G}$ is to consider a least-squares criterion:

$$\epsilon_{\mathrm{LS}} = (S^* W \mathbf{G} \mathbf{y} - \mathbf{y})^* \, \mathbf{C}^{-1} \, (S^* W \mathbf{G} \mathbf{y} - \mathbf{y}). \tag{16}$$

Clearly, ϵ_{LS} is minimized with $\mathbf{G} = \mathbf{G}_{\mathrm{CON}}$ of (3), and is therefore inadmissible.

Tikhonov Regularization: The least-squares strategy does not take advantage of the information $\|x\| \le L$. To take this into account, we may minimize (16) subject to $\|\hat{x}\| \le L$ leading to the Tikhonov estimator. The resulting estimator can be shown to be given by

$$\mathbf{G}_{\mathrm{TIK}} = \begin{matrix} (S^* W)^{-1}, & \|W (S^* W)^{-1} \mathbf{y}\| \le L; \\ (W^* W)^{-1} W^* S \left(\lambda \mathbf{C} + S^* P_{\mathcal{W}} S \right)^{-1}, & \text{otherwise,} \end{matrix} \tag{17}$$

where $\lambda > 0$ is chosen such that $\mathbf{y}^* \mathbf{G}_{\mathrm{TIK}} W^* W \mathbf{G}_{\mathrm{TIK}} \mathbf{y} = L^2$. Evidently, the Tikhonov reconstruction is in general *nonlinear*, and does not have an explicit solution. To obtain a linear reconstruction we may select λ as a constant such that the resulting reconstruction is admissible. The possible values of λ satisfying this condition are given in the following proposition.

Proposition 2 *The Tikhonov reconstruction* $\hat{x}_{\rm TIK} = W\mathbf{G}_{\rm TIK}\mathbf{y}$ *with* $\mathbf{G}_{\rm TIK} = (W^*W)^{-1}W^*S\left(\lambda\mathbf{C} + S^*P_{\mathcal{W}}S\right)^{-1}$ *is admissible on* $\mathcal{U} = \{x \in \mathcal{W} | \|x\| \leq L\}$ *iff* $\lambda \geq m/L^2$.

Note that in general, $\hat{x}_{\rm TIK}$ does not dominate the consistent reconstruction $\hat{x}_{\rm CON}$. Since $\hat{x}_{\rm CON}$ is inadmissible on $\mathcal{U}$, a strictly dominating reconstruction exists. We next consider a minimax MSE approach and show that the resulting reconstruction dominates $\hat{x}_{\rm CON}$ on $\mathcal{U}$.

Minimax MSE Reconstruction: In this approach, we seek the reconstruction $\hat{x}_{\rm MX} = W\mathbf{G}_{\rm MX}\mathbf{y}$ that minimizes the worst-case MSE over all bounded-norm signals, so that $\mathbf{G}_{\rm MX}$ is the solution to

$$\mathbf{G}_{\rm MX} = \arg\min_{\mathbf{G}} \max_{x \in \mathcal{W}, \|x\| \leq L} \epsilon(\mathbf{G}, x). \tag{18}$$

To solve (18) we note that

$$\mathbf{G}_{\rm MX} = \arg\min_{\mathbf{G}} \max_{x \in \mathcal{W}, \|x\| \leq L} \left\{\epsilon(\mathbf{G}, x) - \epsilon(\mathbf{G}_{\rm CON})\right\}, \tag{19}$$

where $\hat{x}_0 = W\mathbf{G}_{\rm CON}\mathbf{y}$ is the consistent reconstruction with $\mathbf{G}_{\rm CON} = (S^*W)^{-1}$. This follows from the fact that, as we have seen in Section 2, $\epsilon(\mathbf{G}_{\rm CON})$ is independent of x. From Theorem 1 we have immediately that $\hat{x}_{\rm MX}$ is admissible, and strictly dominates $\hat{x}_{\rm CON}$ on $\mathcal{U}$. Theorem 3 can now be used to derive an explicit solution to (19), leading to

$$\mathbf{G}_{\rm MX} = \frac{L^2}{L^2 + {\rm Tr}\left((S^*P_{\mathcal{W}}S)^{-1}\mathbf{C}\right)}(S^*W)^{-1}. \tag{20}$$

Note that when $L \to \infty$, we have $\mathbf{G}_{\rm MX} \to \mathbf{G}_{\rm CON}$. Thus, when $x \in \mathcal{W}$ is not norm bounded, the minimax MSE reconstruction coincides with the consistent reconstruction.

5. MSE PERFORMANCE ON THE ENTIRE SPACE

We now consider the case in which x is an arbitrary vector in $\mathcal{W}$.

From Theorem 1 it follows that an admissible reconstruction $\hat{x} = \widehat{\mathbf{G}}\mathbf{y}$ strictly dominating an inadmissible method $\hat{x}_0$ on $\mathcal{W}$ can be constructed as

$$\widehat{\mathbf{G}} = \min_{\mathbf{G}} \max_{x \in \mathcal{W}} \left\{\epsilon(\mathbf{G}, x) - \epsilon(\mathbf{G}_0, x)\right\}. \tag{21}$$

Theorem 5 *Denote the orthogonal projection onto* $\mathcal{N}(P_{\mathcal{W}} - R_0)^{\perp}$ *by* $P_{\mathcal{R}}$ *and define*

$$X = (P_{\mathcal{W}}S\mathbf{C}^{-1}S^*(P_{\mathcal{W}} - R))^{\dagger} - P_{\mathcal{R}}W(W^*S\mathbf{C}^{-1}S^*W)^{-1}W^*P_{\mathcal{R}}.$$

Then the matrix $\widehat{\mathbf{G}}$ *is the solution to (21) iff it satisfies the conditions below with* $R = W\widehat{\mathbf{G}}S^*P_{\mathcal{W}}$ *and* $R_0 = W\mathbf{G}_0S^*P_{\mathcal{W}}$:

1. $\mathbf{C}^{-1}S^*W\widehat{\mathbf{G}} = \widehat{\mathbf{G}}^*W^*S\mathbf{C}^{-1}$;
2. $0 \preceq \mathbf{C}^{-1}S^*W\widehat{\mathbf{G}} \preceq \mathbf{C}^{-1}$;
3. $(P_{\mathcal{W}} - R)P_{\mathcal{R}} = (P_{\mathcal{W}} - R)$;
4. $(P_{\mathcal{W}} - R)^*(P_{\mathcal{W}} - R) \preceq (P_{\mathcal{W}} - R_0)^*(P_{\mathcal{W}} - R_0)$;
5. $(P_{\mathcal{W}} - R)^*(P_{\mathcal{W}} - R)X = (P_{\mathcal{W}} - R_0)^*(P_{\mathcal{W}} - R_0)X$.

An explicit closed form solution for $\widehat{\mathbf{G}}$ can be obtained when $P_{\mathcal{W}}S\mathbf{C}^{-1}S^*P_{\mathcal{W}}$ and $W\mathbf{G}_0S^*P_{\mathcal{W}}$ have the same eigenvectors. For arbitrary choices of $S, W, \mathbf{C}^{-1}$ and $\mathbf{G}_0$ the optimal solution can be obtained using an SDP formulation similar to that presented in the previous section.

Theorem 6 *Let* $P_{\mathcal{W}}S\mathbf{C}^{-1}S^*P_{\mathcal{W}} = U\Sigma U^*$ *where* $\Sigma = diag(\sigma_1, \ldots, \sigma_m)$ *with* $\sigma_i > 0$, *and suppose that* $W\mathbf{G}_0S^*P_{\mathcal{W}} = U\Delta U^*$ *where* $\Delta = diag(\delta_1, \ldots, \delta_m)$. *Then*

$$\widehat{\mathbf{G}} = (W^*W)^{-1}W^*U\mathbf{D}U^*W(S^*W)^{-1}$$

where $\mathbf{D} = diag(d_1, \ldots, d_m)$ *with* $d_i = \max(1 - |1 - \delta_i|, 0)$.

Theorem 5 can be used to develop necessary and sufficient conditions on $\hat{x}_0$ to be admissible for all $x \in \mathcal{W}$:

Theorem 7 *The reconstruction* $\hat{x} = W\widehat{\mathbf{G}}\mathbf{y}$ *is admissible on* $\mathcal{W}$ *iff*

1. $\mathbf{C}^{-1}S^*W\widehat{\mathbf{G}} = \widehat{\mathbf{G}}^*W^*S\mathbf{C}^{-1}$;
2. $0 \preceq \mathbf{C}^{-1}S^*W\widehat{\mathbf{G}} \preceq \mathbf{C}^{-1}$.

A general class of admissible reconstructions is given in the following proposition.

Proposition 3 *Let* $P_{\mathcal{W}}S\mathbf{C}^{-1}S^*P_{\mathcal{W}} = U\Sigma U^*$ *where* $\Sigma = diag(\sigma_1, \ldots, \sigma_m)$ *with* $\sigma_i > 0$. *Then* $\hat{x} = W\widehat{\mathbf{G}}\mathbf{y}$ *with*

$$\widehat{\mathbf{G}} = (W^*W)^{-1}W^*U\mathbf{D}U^*W(S^*W)^{-1} \tag{22}$$

is admissible on $\mathcal{W}$ *where* $\mathbf{D}$ *is a diagonal matrix with diagonal elements* $0 \leq d_i \leq 1$.

An immediate consequence of the proposition is that the consistent reconstruction is admissible on $\mathcal{W}$.

6. FIR RECONSTRUCTION

An interesting class of reconstruction methods is based on processing the samples by a digital correction filter. In this case the noisy samples are convolved with an FIR filter of length $2M + 1$ with coefficients $\{g_i, -M \leq i \leq M\}$ producing the cleaned samples $d_i = y_i * g_i$. The corresponding matrix $\mathbf{G}$ in Fig. 1 is a band Toeplitz convolution matrix with ith diagonal equal to g_i.

For simplicity, we assume that $S^*W = I$ and $\mathbf{C} = \sigma^2\mathbf{I}$ for some $\sigma^2 > 0$. Under this model, our goal is to characterize all admissible FIR filters on $\mathcal{W}$.

From Theorem 7 it follows that the filter g_i is admissible iff the corresponding matrix $\mathbf{G}$ satisfies

$$\mathbf{G} = \mathbf{G}^*; \tag{23}$$

$$0 \preceq \mathbf{G} \preceq \mathbf{I}. \tag{24}$$

Condition (23) implies that the filter must be symmetric, *i.e.*, $g_i = g_{-i}$, so that $\mathbf{G}$ is a symmetric band Toeplitz matrix. Note that such a matrix is completely determined by its first row. To address the condition (24) we rely on the following theorem.

Theorem 8 *Let* $\mathbf{A}$ *be an* $m \times m$ *symmetric Toeplitz matrix with first row equal to* $\{a_i, 0 \leq i \leq m-1\}$. *Then* $\mathbf{A} \succeq 0$ *iff* $A(\omega) \geq 0$ *for all* $\omega \in [0, 2\pi]$, *where*

$$A(\omega) = \sum_{n=-(m-1)}^{m-1} a_n e^{-j\omega n}, \tag{25}$$

is the discrete-time Fourier transform (DTFT) of $\{a_i\}$.

Using the theorem, it is easy to see that (24) is equivalent to

$$0 \leq G(\omega) \leq 1, \quad \forall\omega \in [0, 2\pi]. \tag{26}$$

A sufficient condition for $G(\omega) \leq 1$ is $\sum_{i=-M}^{M} |g_i| \leq 1$.

We conclude that $\hat{x}_0$ is admissible iff g_i is symmetric, and its DTFT satisfies $0 \leq G(\omega) \leq 1$.

6.1. Moving Average Reconstruction

In [3] the authors consider the problem of reconstructing a bandlimited signal from a finite set $m = 2N + 1$ of noisy measurements (see also [5] and reference therein). Their problem can be cast in our general framework by choosing

$$s_i(t) = w_i(t) = \text{sinc}\left(\pi(t - iT)/T\right), \; -N \le i \le N. \quad (27)$$

To reconstruct the signal from the samples they proposed filtering the samples with a symmetric FIR filter with impulse response g_i satisfying $\sum_{i=-M}^{M} |g_i| \le 1$. In particular, they considered in detail the case in which $g_i = 1/(2M + 1)$.

The choice of sampling and reconstruction vectors (27) results in $S^*W = W^*W = I$. It then follows from our discussion that the reconstruction is admissible iff $G(\omega) \ge 0$. Clearly this does not hold for $g_i = 1/(2M + 1)$. Since this filter is inadmissible, there exists a transformation $\mathbf{G}$ that strictly dominates it for all $x \in \mathcal{W}$. To find a strictly dominating $\mathbf{G}$ we now use Theorem 6 where $\mathbf{G}_0$ is the matrix representing convolution with the filter $g_i = 1/(2M + 1), -M \le i \le M$.

It is easy to see that in this case the conditions of Theorem 6 are satisfied. Let $\mathbf{G}_0$ have an eigendecomposition $\mathbf{G}_0 = \mathbf{V}\Delta\mathbf{V}^*$ where $\mathbf{V}$ is a unitary matrix and $\Delta = \text{diag}(\delta_1, \ldots, \delta_m)$. Then a matrix strictly dominating $\mathbf{G}_0$ is given by $\mathbf{G}_0 = \mathbf{V}\mathbf{D}\mathbf{V}^*$, where $\mathbf{D} = \text{diag}(d_1, \ldots, d_m)$ with $d_i = \max(\delta_i, 0)$. Thus, $\mathbf{G}$ is constructed from $\mathbf{G}_0$ by replacing each negative eigenvalue by 0.

To illustrate the performance advantage of $\mathbf{G}_0$, suppose that we are given $m = 2N+1$ noisy samples of the bandlimited signal

$$x(t) = 4\pi\sqrt{2\pi}\frac{\cos^2(t/2)}{\pi^2 - t^2}, \quad (28)$$

at times $t = iT, -N \le i \le N$ where T is the sampling period. The noise is independent and uniform on $[-Z/2, Z/2]$.

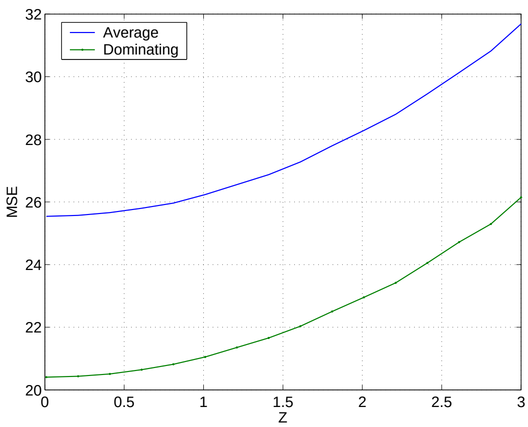

Fig. 2. MSE in estimating $x(t)$ of (28) as a function of Z using the moving average method and the dominating method.

In Fig. 2 we plot the MSE as a function of Z with $m = 75, M = 5, T = 1.5$ using the moving average reconstruction of [3] and our reconstruction. As can be seen from the figure, our approach dominates the moving average method for all Z.

Although our reconstruction dominates the moving average estimator, it no longer corresponds to digital filtering. An important problem therefore that we are currently investigating is to design an FIR filter that leads to a reconstruction strictly dominating a given inadmissible FIR-based method.

7. CONCLUSION

We considered the problem of reconstructing a signal x in a finite-dimensional subspace $\mathcal{W}$ of an arbitrary Hilbert space $\mathcal{H}$ from its nonideal, noisy samples. The samples are modelled as the inner products of x with a set of reconstruction vectors, which span a sampling space $\mathcal{S}$, and recovery is performed in the reconstruction space $\mathcal{W}$ where in general $\mathcal{W}$ and $\mathcal{S}$ can be different, but are assumed to satisfy the condition (2).

We characterized all reconstructions that are MSE admissible, and considered explicit methods for dominating an inadmissable estimator. We also proposed several specific admissible reconstruction methods, by adapting known estimation approaches to our context. In particular, we considered solutions based on digital filtering and showed that admissibility can be determined by examining the frequency response of the filter.

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