OVERSAMPLING OF THE GENERALIZED MULTIWINDOW GABOR SCHEME

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ABSTRACT

A class of non-canonical duals for the multiwindow Gabor scheme, incorporating both rational and integer oversampling of the Gaborian combined time-frequency space, is considered. Using properties of Gabor frame matrices, block Discrete Fourier transforms, and results from number theory, we establish existence conditions for the non-canonical duals for both integer and rational oversampling rates, in the signal domain. For comparison and completeness of the results, we also obtain the equivalent results in the Zak transform domain. We show that, using this approach, both computational efficiency and stability of the Gabor scheme can be enhanced.

1. INTRODUCTION

Multiwindow Gabor expansion combines the advantages of localization in the combined time-frequency space, characteristic of the classical Gabor scheme, as well as scale-space properties of wavelets. Multiwindow Gabor expansions of signals (and images) find many applications in the fields of pattern recognition, computer vision, and recently macro-molecular sequence analysis.

In [1], Zibulski and Zeevi introduced multiwindow Gabor expansions and in [2], extended the concept to the finite, discrete-time case. The coefficients of the multiwindow Gabor expansion are given by the projection of the finite signal $\mathbf{f} \in \mathcal{C}^L$ onto the combined space

$$c_{r,m,n} = \sum_{k=0}^{L-1} f[k] g_r[k - na] e^{-j2\pi mbk/L}, \qquad (1)$$

where $g_r[k], r \in 0, \ldots, R-1$ are the window functions, a and b are the combined space sampling intervals along the time and frequency axes respectively.

Given the coefficients $c_{r,m,n}$, the analysis windows $g_r[k]$, and the lattice constants a and b, we wish to reconstruct the signal $f[k]$ from the coefficients. The reconstruction of the signal $f[k]$ is given by [2]

$$f[k] = \sum_{r=0}^{R-1} \sum_{m=0}^{\overline{b}-1} \sum_{n=0}^{\overline{a}-1} c_{r,m,n} \gamma_r[k - na] e^{j2\pi mbk/L}, \qquad (2)$$

where $\gamma_r[k]$ are the dual windows, $\overline{a} = L/a \in \mathcal{N}$ and $\overline{b} = L/b \in \mathcal{N}$ are the number of sampling intervals along the time and frequency axes respectively. We assume L is divisible by both a and b [5].

In vector form, (1) can be written as

$$\mathbf{c} = \mathbf{G}^* \mathbf{f}, \qquad (3)$$

where $\mathbf{c}$ is the vector of coefficients, and $\mathbf{G}$ is the Gabor matrix

$$\mathbf{G} = \begin{bmatrix} g_{0,0,0}[0] & \cdots & g_{R-1,\overline{a}-1,\overline{b}-1}[0] \\ g_{0,0,0}[1] & \cdots & g_{R-1,\overline{a}-1,\overline{b}-1}[1] \\ \vdots & \ddots & \vdots \\ g_{0,0,0}[L-1] & \cdots & g_{R-1,\overline{a}-1,\overline{b}-1}[L-1] \end{bmatrix} \qquad (4)$$

with $g_{r,m,n}[k] = g_r[k - na] e^{j2\pi mbk/L}$. The reconstruction, inverse of (1), is the following vector form of (2):

$$\mathbf{f} = \mathbf{\Gamma} \mathbf{c}, \qquad (5)$$

where $\mathbf{\Gamma}$ is the dual of the Gabor matrix.

It was established in [2] that a necessary condition for complete reconstruction in the case of multiwindow Gabor expansions is given by $R\overline{a}\overline{b} \geq L$. In the case of critical sampling and a single window, the reconstruction is unstable according to the Balian-Low theorem [6]. This theorem extends to well-behaved multiwindows [1]. We, therefore, consider only the oversampling case where $R\overline{a}\overline{b} > L$, which implies that the functions $g_{r,m,n}[k]$ are linearly dependent and the representation is overcomplete.

As the representation is overcomplete, there exist an infinite number of possible duals $\gamma_r[k]$. The canonical solution is to find the minimum norm dual of the set of generalized Gabor elementary functions $g_{r,m,n}[k]$ by [4], [5],

$$\tilde{\gamma}_r[k] = (\mathbf{G}\mathbf{G}^*)^{-1} g_r[k]. \qquad (6)$$

However, it is often better to choose a different dual from a wider set of duals, as we show in Section 4. Here, we extend the non-canonical duals introduced in [3], and applied to single window Gabor expansions in [9] to multiwindow Gabor expansions.

The non-canonical dual [3] is given by

$$d_{r,m,n} = d_r[k-na]e^{j2\pi mbk/L} = (\mathbf{H}\mathbf{G}^*)^{-1}h_r[k], \quad (7)$$

where $\mathbf{H}$ is another Gabor matrix of the same form as $\mathbf{G}$ such that $\mathbf{H}\mathbf{G}^*$ is invertible. We obtain existence conditions for non-canonical multiwindow Gabor frames, both in the general context and in integer oversampling and rational oversampling cases. For comparison, we provide the equivalent results in the finite Zak transform domain and discuss some advantages of non-canonical duals.

The paper is organized as follows: In Section 2, we obtain the existence conditions for the non-canonical multiwindow Gabor dual in the general case. In Section 3, we consider the conditions for both integer, and rational oversampling cases, and also discuss the existence condition in the Zak transform domain. In Section 4, we show examples of non-canonical duals, and discuss their advantages.

2. EXISTENCE CONDITIONS FOR NON-CANONICAL DUALS

The general condition satisfied by all duals, (called the Wexler-Raz condition in the single window Gabor case) can be extended to the case of multiwindow Gabor expansion, the proof of which can be found in [8]:

Theorem 2.1. *The Wexler-Raz identity [4] in the multiwindow case states that, for every set of dual functions* $\gamma_r, r \in 0,\ldots,R-1$*, the following identity is satisfied.*

$$\sum_{r=0}^{R-1}\sum_{k=0}^{L-1} g_r{}^*[k-n\overline{b}]e^{-j2\pi mk/a}\gamma_r[k] = (L/\overline{a}\overline{b})\delta_m\delta_n, \quad (8)$$

where $n \in 0,\ldots,b-1, m \in 0,\ldots,a-1$.

From [9], we know that the condition for the existence of the dual frame is equivalent to the invertibility of the frame matrix $\mathbf{P} = \mathbf{H}\mathbf{G}^*$. In [8], we show that the matrix $\mathbf{P} = \mathbf{H}\mathbf{G}^*$ is a block circulant matrix [10], where each block is of size $a \times a$. Using block circulant properties of frame $\mathbf{P}$ leads to the following condition of invertibility:

Theorem 2.2. *[10] The matrix* $\mathbf{P}$ *is invertible if and only if all the* $a \times a$ *blocks of the block discrete Fourier transform (BDFT) of* $\mathbf{P}$ *are invertible individually.*

The BDFT of a block circulant matrix $\mathbf{P} = \mathcal{C}(A_0, A_1, \ldots, A_{\overline{a}-1})$ is given by $\mathcal{F}\mathbf{P} = \mathcal{C}(\hat{\mathbf{A}}_\mathbf{0}), \hat{\mathbf{A}}_\mathbf{1}, \ldots, \hat{\mathbf{A}}_{\overline{\mathbf{a}}-\mathbf{1}})$, where $\mathcal{C}$ stands for circulation of the blocks in the block circulant matrix $\mathbf{P}$ and

$$\hat{\mathbf{A}}_q = \sum_{p=0}^{\overline{a}-1}\omega^{pq}\mathbf{A}_p, \; 0 \le q \le \overline{a}-1, \quad (9)$$

where $\omega = e^{-j2\pi/\overline{a}}$.

Apart from these sufficient conditions for the existence of the dual, there is a necessary condition that helps in choosing the synthesis function.

Theorem 2.3. *[8] For the matrix* $\mathbf{P}$ *to be invertible, the condition given below must be satisfied.*

$$\sum_{r=0}^{R-1}\sum_{n=0}^{\overline{a}-1} h_r[k-na]g_r{}^*[k-na+q\overline{b}] \neq 0, \quad (10)$$

for all $k \in 0,\ldots,L-1, q \in 0,\ldots,b-1$.

From (10), we can show that the spread of the window functions g_r and h_r should be at least a [8]. This can be seen as a sort of discrete equivalent of Daubechies' result [6] in the continuous case $\underset{x\in[0,q_0]}{\text{ess inf}} \sum_n \mid g(x-nq_0) \mid^2 > 0$.

3. SPECIFIC CASES AND EXISTENCE CONDITIONS

3.1. Integer Oversampling

Using the condition given by Theorem 2.2 and the property of the banded structure of the Gabor matrix [5], we obtain a stronger (and more easily verifiable) condition on the invertibility of the matrix $\mathbf{P}$ in the case of integer oversampling.

Theorem 3.1. *For the matrix* $\mathbf{P}$ *to be invertible in the integer oversampling case, the following equation must be satisfied.*

$$\sum_{q=0}^{b-1}\sum_{v=0}^{a-1}\sum_{r=0}^{R-1}\langle h_r[u], g_r[u+q\overline{b}]e^{j2\pi vu/a}\rangle e^{j2\pi sq/b}e^{j2\pi vk/a} \neq 0, \quad (11)$$

for any $k, s \in 0,\ldots,\overline{b}-1$.

Proof. We begin with the properties of block circulant matrix. The BDFT can be written as

$$\tilde{A_s} = A_0 + e^{j2\pi s/\overline{b}}A_1 + \cdots + e^{j2\pi s(b-1)\overline{b}s/a\overline{a}}A_{(b-1)\overline{b}/a} \quad (12)$$

The diagonal elements of $\tilde{A_s}$ (the only non zero elements as shown in [8]) are given by

$$A_s(\tilde{k},k) = \overline{b}\sum_{r=0}^{R-1}\sum_{n=0}^{\overline{a}-1} h_r[k-na]\sum_{q=0}^{b-1} g_r{}^*[k-na-q\overline{b}]e^{j2\pi qs/b}$$

$$= \overline{b} \sum_{q=0}^{b-1} e^{j2\pi qs/b} \sum_{r=0}^{R-1} h_r[k] {g_r}^*[k - q\overline{b}] \sum_{n=0}^{\overline{a}-1} \delta[k - na]$$

$$= \frac{\overline{b}}{a} \sum_{q=0}^{b-1} e^{j2\pi qs/b} \sum_{r=0}^{R-1} h_r[k] {g_r}^*[k - q\overline{b}] \sum_{v=0}^{a-1} e^{j2\pi vk/a}$$

$$= \frac{\overline{b}}{a} \sum_{q=0}^{b-1} \sum_{v=0}^{a-1} e^{j2\pi qs/b + (k-u)v/a} \sum_{r=0}^{R-1} \sum_{u=0}^{L-1} h_r[u] \, {g_r}^*[u - q\overline{b}]$$

$$= \frac{\overline{b}}{a} \sum_{q=0}^{b-1} \sum_{r=0}^{R-1} \sum_{u=0}^{L-1} h_r[u] {g_r}^*[u - q\overline{b}] e^{j2\pi qs/b} \sum_{v=0}^{a-1} e^{-j2\pi(vu/a - kv/a)}. \qquad (13)$$

Equation (13) gives the condition we mentioned in the theorem and thus the theorem is proved. □

Theorem 3.1 can be reworded by requiring that the BDFT of $\mathbf{P} = \mathbf{HG}^*$ have no zeroes along its principal or $l\overline{b}$-th subdiagonals.

From Theorem 3.1, we can develop another condition that explicitly permits certain types of functions to generate frames for $\mathcal{C}^L$.

Theorem 3.2. *A sufficient condition for the invertibility of $\mathbf{HG}^*$ is that the sequences $g_r[k - na - q\overline{b}], q \in 0, \ldots, b-1$ be positive (or negative) definite, when $h_r[k]$ are all of the same sign.*

Proof. From the properties of the block circulant matrices (12), we have the values of BDFT of $\mathbf{P}$. Since $\overline{b}$ is divisible by a, the matrix is block circulant in blocks of $\overline{b}$ as well as in a [8]. We can see that the Fourier matrices $\tilde{A_s}$ will be diagonal as well. If we can show that the diagonal elements of the Fourier block matrix $\tilde{A_s}$ are not zero, then the matrix will always be invertible and the proof is complete. The diagonal elements of $\tilde{A_s}$ are given by

$$A_s(\tilde{k}, k) \qquad (14)$$

$$= \overline{b} \sum_{r=0}^{R-1} \sum_{q=0}^{b-1} \sum_{n=0}^{\overline{a}-1} {g_r}^*[k - na - q\overline{b}] h_r[k - na] e^{j2\pi qs/b}$$

$$= \overline{b} \sum_{r=0}^{R-1} \sum_{n=0}^{\overline{a}-1} h_r[k - na] \sum_{q=0}^{b-1} {g_r}^*[k - na - q\overline{b}] e^{j2\pi qs/b}$$

$$= \overline{b} \sum_{r=0}^{R-1} \sum_{n=0}^{\overline{a}-1} h_r[k - na] \hat{g_r}[k - na - q\overline{b}], \qquad (15)$$

where $\hat{g_r}[k - na - q\overline{b}] = \sum_{q=0}^{b-1} {g_r}^*[k - na - q\overline{b}] e^{j2\pi qs/b}$. Now it can be easily seen that $\hat{g_r}[k - na - q\overline{b}]$ is the b-point DFT of the b point sequence ${g_r}^*[k - na - q\overline{b}]$. If the DFT of the sequence is positive and all the individual components $h_r[k]$ are themselves positive everywhere, then $\tilde{A_s}(k, k)$ is real and positive. Since these values correspond to the values of the diagonal on the $\overline{b} \times \overline{b}$ diagonal matrix, the matrix will always be invertible. Similar arguments hold for the negative definite case.

It can easily be seen that (15) cannot be zero if the conditions are satisfied. Therefore, the statement is proved. □

3.2. Rational Oversampling

Theorem 2.2 requires us to prove that the $a \times a$ BDFT submatrices are invertible for the matrix $\mathbf{P}$ to be invertible. However, in the case of rational oversampling, these submatrices have a very definite structure. The structure of the BDFT submatrices are given by the following theorem.

Theorem 3.3. *Let the greatest common divisor (gcd) of $(a, \overline{b}) = \alpha$. Then the non-zero elements in the BDFT of the matrix $\mathbf{P}$ are at least at a distance α from each other on each row.*

Proof. Consider the first block $a \times L$ of the matrix $\mathbf{P}$ which we use in the generation of the block circulant Fourier matrix. It is easily apparent that structurally (the position of zeroes to the non-zero elements), the latter rows are simply the first row shifted right by the appropriate distance from the first row. Therefore, considering only the first row alone in the block circulant matrix $\mathbf{P}$, we find the block Fourier transform of this row.

In the first row, we have the non zero elements at positions $0, \overline{b}, \ldots, \overline{b}(b-1)$. We add the corresponding elements of the $a \times a$ matrices after the multiplication with the appropriate $e^{-j2\pi mk/L}$, where $m, k \in 0, \ldots, \overline{a} - 1$.

The elements of the first row of the $a \times a$ matrix can be written as $\beta a + \nu$, where $\nu \in 0, \ldots, a-1, \beta \in 0, \ldots, \overline{a}-1$. The non zero elements being at $\mu\overline{b}$, where $\mu \in 0, \ldots, b-1$, the only non zero elements will occur when $\nu + \beta a = \mu\overline{b}$, for some $\beta \in 0, \ldots, \overline{a} - 1$ and $\mu \in 0, \ldots, b - 1$. But from Euclid's algorithm, we know that the smallest positive number is $\alpha = gcd(a, \overline{b})$. Therefore, ν cannot be less than α. Thus the non zero elements of the block Fourier matrix of $\mathbf{P}$ have to be at least α apart. □

Utilizing perfect shuffle matrices [12], matrices $\tilde{\mathbf{A}}_k, k \in 0, \ldots, \overline{a} - 1$, can be factorized as shown in Theorem 3.4 to block diagonal matrices of size $\frac{a}{\alpha} \times \frac{a}{\alpha}$. Of course, in the worst case $\alpha = 1$, we have the same as the previous problem, i.e., showing that an $a \times a$ matrix has to be invertible. Otherwise, we can show that the α matrices of size $\frac{a}{\alpha} \times \frac{a}{\alpha}$ are invertible in place of showing that an $a \times a$ matrix is invertible leading to a simplification of the size of the problem.

Theorem 3.4. *[12] An $a \times a$ matrix which has non-zero elements on the principal diagonal and at a distance of $k\alpha, k \in 1, \ldots, \frac{a}{\alpha} - 1$ from the diagonal can be factorized into α block diagonal matrices of size $\frac{a}{\alpha} \times \frac{a}{\alpha}$ using a perfect shuffle matrix $\mathbf{V}_{\alpha,a}$. The block diagonal matrix $\mathbf{W}$ is created using the formula*

$$\mathbf{W} = \mathbf{V}^*{}_{\alpha,a}\tilde{\mathbf{A}}_s\mathbf{V}_{\alpha,a}, \tag{16}$$

where $\tilde{\mathbf{A}}_s$ is the s-th block of the BDFT of the matrix $\mathbf{P}$.

3.3. Zak Transform Domain Results

The finite Zak transform (FZT), of a function $\mathbf{f} \in \mathcal{C}^L$ denoted by $Z_{\overline{b}}$ is defined as the mapping $Z_{\overline{b}} : \mathcal{C}^L \to \mathcal{C}^{\overline{b}} \times \mathcal{C}^b$, given by the equation

$$(Z_{\overline{b}})(r, v) = \sum_{k=0}^{b-1} f(r - \overline{b}k)e^{\frac{j2\pi\overline{b}kv}{L}}, \tag{17}$$

where $\overline{b} \in \mathcal{N}$ is a fixed parameter.

Let $\frac{ab}{L} = \frac{p}{q}$, where p and q are mutually prime. Based on the definition of FZT, we define piecewise Finite Zak Transform (PFZT) as a vector valued function of size p:

$$\mathbf{F}(r, v) = \left[F_0(r, v), \ldots, F_{p-1}(r, v)\right]^T, \tag{18}$$

where

$$F_l(r, v) \stackrel{\triangle}{=} (Zf)\left(r, v + l\frac{b}{p}\right), \quad 0, \ldots, p-1. \tag{19}$$

It is important to note that inner products are preserved across PFZTs.

Using these results, we can define the action of the frame operator in the Zak transform domain, by [11]

$$(\mathcal{P}\mathbf{F})(r, v) = \mathcal{P}(r, v)\mathbf{F}(r, v), \tag{20}$$

where both $\mathcal{P}$ and $\mathbf{F}$ are the PFZTs of the frame operator $\mathbf{P}$ and $\mathbf{f}$ respectively. The elements of the $p \times p$ matrix constituents of the PFZT of $\mathcal{P}$ are given by

$$\begin{aligned}\mathcal{P}_{k,l}(r, v) \quad = \quad & \frac{\overline{b}}{p}\sum_{s=0}^{q-1}(Zh)(r - sa, v + kb/p) \\ & (Zg)^*(r - na, v + lb/p)\end{aligned} \tag{21}$$

and $(Zg)^*(r, v)$ is given by (17).

From the result in [11], we obtain the condition for the existence of the Gabor frame in the general case:

Theorem 3.5. *Given that $g, h \in l^2(\mathcal{Z}/L), \overline{a}\overline{b} \geq L$, and a matrix valued function as shown in equation (21), the matrix $\mathbf{HG}^*$ constitutes a frame operator if and only if $det(\mathcal{P})(r, v) \neq 0$ for all $r, v \in \mathcal{Z}^2$.*

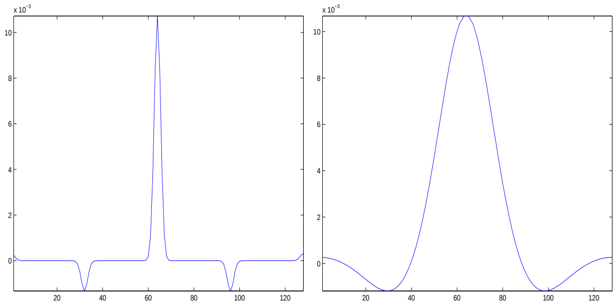

Fig. 1. The canonical dual for the multiple Gaussian windows with lattice constants ($a = 2, b = 4, \sigma_n g = \sigma_n h = 2, \sigma_w g = \sigma_w h = 16, L = 128$).

It is interesting to observe the similarities between the Zak transform domain methods and the block circulant matrix methods which we have used in the paper previously.

For the case of critical sampling, we have the interesting result that each of the $\mathbf{P}(s, v)$ turns into a scalar value function and we have the result that each of the $\mathbf{P}(s, v)$ should be non zero for all values of s, v - paralleling the result in the canonical dual case. This provides us the invertibility condition for the critical sampling case.

Similarly, for integer oversampling, we have $p = 1$, and therefore, (21), becomes

$$\mathbf{P}(s, v) = \overline{b}\sum_{w=0}^{q-1}\langle(Zh)(s - wa, v), (Zg)^*(s - wa, v)\rangle. \tag{22}$$

As long as (22) is not zero, we have a simple way of determining the invertibility of the frame operators.

4. RESULTS AND DISCUSSION

The technique used to invert the multiwindow Gabor frame operators is the same as the one used for the single window Gabor frame operators. In [8], the authors established an efficient technique to invert the Gabor frame operators in case of integer oversampling. For the case of rational oversampling as well, we have a technique that is slightly better than the existing one in many cases, and at least as good as the existing one in all cases. The computational complexity of finding the dual of the frame is computed to be $O(nlog(n))$ [7] using the conjugate gradients techniques. The factorization technique we introduced for rational oversampling cases can be used to simplify the matrix that has to be inverted.

For the case of integer oversampling, the computational complexity can be easily computed. The number of operations necessary to generate the matrix $\mathbf{S}$ in the case of the minimum norm dual is given by $R(\overline{a} + 1)ab$ multipli-

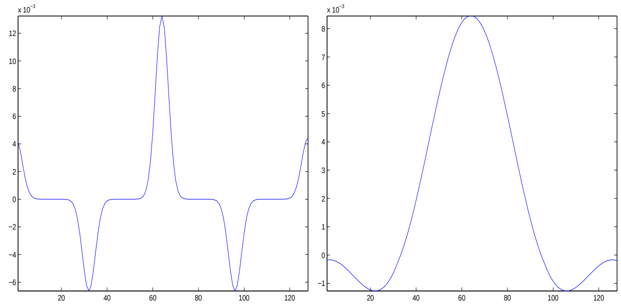

Fig. 2. The non canonical dual for the multiple Gaussian windows with lattice constants ($a = 2, b = 4, \sigma_n g = 2, \sigma_n h = 4, \sigma_w g = 8, \sigma_w h = 16, L = 128$).

cations and $Rb\overline{a}a$ additions [5]. The computational complexity of inverting $\mathbf{S}$ is given by $O(2Llog(\overline{a}))$ [5] and $O(cb(\frac{b^2}{d^2} + blogd))$ [12]. Usually $\overline{a}$ is much larger than all other constants in the lattice - the shift needs to be small enough to capture all the details of the signal. Therefore, anything that eliminates $\overline{a}$ from the computational complexity is going to be much better than the other methods.

In our case, the number of operations necessary to generate the matrix $\mathbf{P}$ is $2Rab$ multiplications and 0 additions. The complexity of computing the Fourier components and inverting P is given by $O(2Llog(b))$. The overall complexity of our algorithm is thus seen to be better than the traditional methods for both integer and rational oversampling.

As we see in Figs. 1 and 2, both the duals retain the localization properties. The non-canonical dual does not change much, especially in the wide window case. The parameters are considerably different, as shown in the figures. It is also worth mentioning that the ratio of the condition numbers of the non-canonical dual to the condition number of the canonical dual is 0.23. This demonstrates that non-canonical duals may have greater stability than canonical duals, under certain conditions.

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