“Beat-Your-Rival” Routing Games

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Abstract. In the traditional setting of routing games, the standard assumption is that selfish agents are unconcerned with the performance of their competitors in the network. We propose an extension to this setting by modeling agents to consider a combination of their own performance as well as that of their *rivals*. Per agent, we parameterize this trade-off, thereby allowing agents to be partially selfish and partially malicious.

We consider two types of routing games based on the structure of the agents' performance objectives, namely *bottleneck routing games* and *additive routing games*. For *bottleneck routing games*, the performance of an agent is determined by its worst-case link performance, and for *additive routing games*, performance is determined by the sum of its link performances. For the bottleneck routing scenario we establish the existence of a Nash equilibrium and show that the *Price of Stability* is equal to 1. We also prove that the *Price of Anarchy* is unbounded. For *additive routing games*, we focus on the fundamental load balancing game of routing over parallel links. For an interesting class of agents, we prove the existence of a Nash equilibrium. Specifically, we establish that a special case of the *Wardrop equilibrium* is likewise a *Nash equilibrium*. Moreover, when the system consists of two agents, this Nash equilibrium is unique, and for the general case of N agents, we present an example of its non-uniqueness.

1 Introduction

To date, game theoretic models have been employed in virtually all networking contexts. These include control tasks at the network layer, such as flow control (e.g., [16]), and routing (e.g., [1, 5, 18, 20, 24, 25] and references therein), as well as numerous studies on control tasks at the link and MAC layers. A fundamental assumption in all of these referenced studies is that the selfish agents compete over resources in the network and aim to optimize their own performance; agents do not care (either way) about the performance of their competitors. However, and typically in the context of routing, scenarios exist in which this assumption is not warranted.

For example, consider the scenario where two Content Providers, A and B, offer video-on-demand services in a network. Both A and B compete over the network resources, however only Content Provider A aspires to minimize its own latency. Due to business considerations, Content Provider B aims at offering its clients a performance that is equal or better than A's performance. Thus, the objective of B is not solely to maximize its performance.

In light of examples like the one above, previous research in routing games has extended the classical model of “performance-maximizing” or “selfish” agents, and focused on different scenarios, e.g., settings where certain agents may act *maliciously* towards other agents [4, 7, 22]. Such malicious behavior could be due to a range of reasons, e.g., hackers or rivaling companies that aim to degrade network quality. In contrast, other studies in routing games consider agents to have an altruistic component to their objective [3, 9, 15].

In order to best model real-life scenarios, each agent's objective should lie somewhere in the range between *malicious*, *selfish* and *altruistic*, as depicted in Figure 1. This direction has been proposed in [11], where each agent i has a parameter that captures how important the social performance is to i. In this setting, a malicious agent aims to minimize the social performance, an altruistic agent aims to maximize it and a selfish agent does not take the social performance into account at all. However, [11] focuses on a non-atomic game, i.e., a game with an infinite amount of agents, where each agent controls a negligible amount of flow. Following a similar course, in [3, 9, 10, 15], agents are of finite size, and their objectives are parameterized to lie somewhere between *selfish* and *altruistic*, yet *malicious* objectives are not taken into account.

In this study, we intend to investigate agents of finite size whose objectives lie in the range between *malicious* and *selfish*. Per agent i, we parameterize this trade-off through a coefficient $\alpha^i \in [0, 1]$, where $\alpha^i = 1$ corresponds to a selfish agent and $\alpha^i = 0$ to a malicious agent. However, unlike [11], we represent agent i's cost as a combination of its own performance and that of its *rival*. We define the rival of an agent i as the agent $j \neq i$ with the current best performance in the system. Note that an agent's rival is not fixed, but is dependent on the current performance of all the agents in the system. In our setting, a totally malicious agent aims to minimize the performance of its rival, while a totally selfish agent does not take its rival's performance into account.

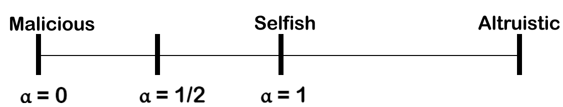

Fig. 1. The range of agents' objectives.

We consider two types of routing games based on the structure of the agents' performance objectives. The first game considers agents with *bottleneck objectives* (also known as Max-Min or Min-Max objectives), i.e., their performance is determined by the worst component (link) in the network [5,8,12]. *Bottleneck routing games* have been shown to emerge in many practical scenarios. For example, in wireless networks, the weakest link in a transmission is determined by the node with the least remaining battery power. Hence, each agent would route traffic so as to maximize the smallest battery lifetime along its routing topology. Additionally, bottleneck routing games arise in congested networks where it is desirable to move traffic away from congested hot spots. For further discussion and additional examples see [5]. The second type of game considers agents with *additive* performance measures, e.g., delay or packet loss. Much of the current literature on networking games has focused on such games, e.g., [1, 14, 17–20, 25], albeit in the traditional setting of selfish agents.

In [5] and [20], the existence of a Nash equilibrium has been established respectively, for bottleneck and additive routing games with selfish agents. We note that a major complication in proving the existence of a Nash equilibrium for agents with a malicious component, i.e, $\alpha^i < 1$, is the inherent lack of convexity of the objective functions. Thus, we cannot rely on the proofs of existence from the referenced works, and need to establish proofs of our own that do not require (quasi-)convexity of the performance functions.

For both types of games, many studies have attempted to bound the Price of Anarchy (PoA) [17] and the Price of Stability (PoS) [2]. The PoA and PoS quantify the deficiency of the network from a social perspective, at respectively, the worst and best Nash equilibrium. Due to the ever-growing work in this context, it is beyond the scope of this writing to do justice and present an exhaustive survey of previous work on routing games with selfish agents. We refer the reader to the above cited papers and to the references therein for a broader review of the literature.

1.1 Our Contribution

We focus our study on the atomic splittable routing model [5, 20], in which each agent sends its non-negligible demand to its destination by splitting it over a set of paths in the network. All agents share the same source and destination, and each agent i has a coefficient α^i, which captures the importance of its rival's performance. We first consider agents with *bottleneck performance measures*, and for which $\alpha^i \in [1/2, 1]$. Intuitively, this range of α^i implies that they care more about their own performance than that of their rivals'. We prove that the Price of Stability is equal to 1, i.e., there always exists a system optimal Nash equilibrium. Moreover, we establish that the Price of Anarchy is unbounded.

We then consider agents with *additive* performance objectives and focus on the fundamental load balancing game of routing over parallel links. Beyond being a basic framework of routing, this is the *generic framework of load balancing* among servers in a network. It has been the subject of numerous studies in the context of non-cooperative networking games, e.g., [14, 17, 18, 20, 23, 26], to name a few. We consider agents that view their own performance and that of their rivals with equal importance, i.e., for all i, $\alpha^i = 1/2$. We establish the existence of a Nash equilibrium and show that the Wardrop equilibrium (which necessarily exists and is unique [13]) is also a Nash equilibrium. Moreover, for a system with two agents, we prove the Nash equilibrium's uniqueness and for the general case of N agents, we provide an example of its non-uniqueness. Finally, we present an example of a system with agents for which $\alpha^i \in [0, 1]$ and show that for both *bottleneck* and *additive* routing games, no Nash equilibrium necessarily exists.

2 Model and Game Theoretic Formulations

2.1 Model

We consider a set $\mathcal{N} = \{1, 2, \ldots, N\}$ of selfish "users" (or, "players", "agents"), which share a communication network modeled by a directed graph $G(V, E)$. We denote by

$\mathcal{P}$ the set of all paths in the network. Each user $i \in \mathcal{N}$ has a traffic demand r^i and all users share a common source S and common destination T. Denote the total demand of all the users by R, i.e., $R = \sum_{i \in \mathcal{N}} r^i$. For every i, we denote by $-i$ the set of all users in the system, excluding i. A user ships its demand from S to T by splitting it along the paths in $\mathcal{P}$, i.e., user i decides what fraction of r^i should be sent on through each path. We denote by f_p^i, the flow that user $i \in \mathcal{N}$ sends on path $p \in \mathcal{P}$. User i can fix any value for f_p^i, as long as $f_p^i \geq 0$ (non-negativity constraint) and $\sum_{p \in \mathcal{P}} f_p^i = r^i$ (demand constraint); this assignment of traffic to paths, $\mathbf{f^i} = \{f_p^i\}_{p \in \mathcal{P}}$ shall also be referred to as the *routing strategy* of user i. The *(routing strategy) profile* $\mathbf{f}$ is the vector of all user routing strategies, $\mathbf{f} = (\mathbf{f^1}, \mathbf{f^2}, \ldots, \mathbf{f^N})$. We say that a profile $\mathbf{f}$ is feasible if it is composed of feasible routing strategies and we denote by $\mathbf{F}$ the set of all feasible profiles. Turning our attention to a path $p \in \mathcal{P}$, let f_p be the total flow on that path i.e., $f_p = \sum_{i \in \mathcal{N}} f_p^i$; also denote by f_e^i the flow that i sends on link $e \in E$, i.e., $f_e^i = \sum_{p | e \in p} f_p^i$. Similarly, the total flow on link $e \in E$ is denoted by $f_e = \sum_{i \in \mathcal{N}} f_e^i$. We associate with each link a performance function $T_e(\cdot)$, which corresponds to the *cost per unit of flow* through link e and only depends on the total flow f_e. Furthermore, we impose the following assumptions on $T_e(f_e)$:

A1 $T_e : [0, \infty) \rightarrow [0, \infty]$.
A2 $T_e(f_e)$ is continuous and strictly increasing in f_e.

The performance measure of a user $i \in \mathcal{N}$ is given by a cost function $H^i(\mathbf{f})$, which we shall refer to as the *selfish* cost of i. In bottleneck routing games, $H^i(\mathbf{f})$ corresponds to the performance of the worst-case link, and in additive routing games it corresponds to the sum of all link performances in the system. We define the *rival* of i at $\mathbf{f}$, as the user with the lowest selfish cost at $\mathbf{f}$, i.e., $\min_{j \neq i} H^j(\mathbf{f})$. The aim of each user is to minimize the weighted difference between its own cost and the cost of its *rival* in the network. Thus, the aim of i is to minimize

$$J^i(\mathbf{f}) \equiv \alpha^i H^i(\mathbf{f}) - (1 - \alpha^i) \min_{j \neq i} \{H^j(\mathbf{f})\}. \tag{1}$$

Note that $J^i(\mathbf{f})$ is not necessarily convex in its user flows.

2.2 Bottleneck routing cost function

Following [5], we define the bottleneck of a user $i \in \mathcal{N}$, $b^i(\mathbf{f})$, as the worst performance of any link in the network that i sends a positive amount of flow on,

$$b^i(\mathbf{f}) = \max_{e \in E | f_e^i > 0} T_e(f_e).$$

The *selfish cost* of user i is equal to its bottleneck, $H^i(\mathbf{f}) = b^i(\mathbf{f}) = \max_{e \in E | f_e^i > 0} T_e(f_e)$. Thus, we consider users whose cost functions contain the following form,

$$J^i(\mathbf{f}) = \alpha^i \max_{e \in E | f_e^i > 0} \{T_e(f_e)\} - (1 - \alpha^i) \min_{j \neq i} \max_{l \in E | f_l^j > 0} \{T_l(f_l)\}. \tag{2}$$

In other words, user i aims to minimize the weighted difference between its bottleneck and that of its best-off competitor. We define the bottleneck of a path $p \in \mathcal{P}$ with $f_p > 0$ as $b_p(\mathbf{f}) = \max_{e \in p} T_e(f_e)$ and we define the bottleneck of the system as

$$b(\mathbf{f}) = \max_{e \in E | f_e > 0} T_e(f_e).$$

We equate the "welfare" of the system to its bottleneck and denote by $\mathbf{f}^* = (\mathbf{f}^*)_{e \in E}$, the optimal vector of link flows. Thus, the social optimum equals $b(\mathbf{f}^*) = \min_{\mathbf{f} \in \mathbf{F}} b(\mathbf{f})$.

2.3 Additive routing cost functions

An important class of problems is when users are interested additive performance measures, e.g., delay or packet loss. In this case, T_e may correspond to the total delay of link e. For *additive routing games*, we consider the framework of routing in a "parallel links" network. Thus, $G(V, E)$ corresponds to a graph with parallel "links" (e.g., communication links, servers, etc.) $\mathcal{L} = \{1, 2, \ldots, L\}$, $L > 1$, and a users ships its demand by splitting it over the links $\mathcal{L}$. In particular, we consider users whose selfish cost functions are of the following form:

$$H^i(\mathbf{f}) = \frac{1}{r^i} \sum_{l \in \mathcal{L}} f_l^i T_l(f_l). \tag{3}$$

Thus, $H^i(\mathbf{f})$ corresponds to the average sum of the link costs. From (1) we get that

$$J^i(\mathbf{f}) = \alpha^i \sum_{l \in \mathcal{L}} \frac{f_l^i}{r^i} T_l(f_l) - (1 - \alpha^i) \min_{j \neq i} \left\{ \sum_{l \in \mathcal{L}} \frac{f_l^j}{r^j} T_l(f_l) \right\}. \tag{4}$$

2.4 Nash Equilibrium

A profile $\mathbf{f}$ is said to be a Nash equilibrium if, given $\mathbf{f^{-i}}$, no user finds it beneficial to deviate from its routing strategy $\mathbf{f^i}$. More formally, $\mathbf{f}$ is a Nash equilibrium if, for all $i \in \mathcal{N}$ and any feasible routing strategy $\overline{\mathbf{f}}^{\mathbf{i}} \neq \mathbf{f^i}$, the following condition holds

$$J^i(\mathbf{f^i}, \mathbf{f^{-i}}) \leq J^i(\overline{\mathbf{f}}^{\mathbf{i}}, \mathbf{f^{-i}}). \tag{5}$$

In order to quantify the degradation of a Nash equilibrium, we turn towards the *Price of Anarchy* [17] (the *Price of Stability* [2]), which is defined as the ratio between the *worst* (*best*) Nash equilibrium, and the social optimum,

3 Bottleneck Routing Games

We start by establishing the existence of a Nash equilibrium in our bottleneck routing game. Note that the user cost function in (2) is not continuous, as pointed out in [5]. Moreover, $J^i(\mathbf{f})$ is not necessarily quasi-convex in f_l^i. Consequently, we need to construct an existence proof that does not rely on the continuity or the quasi-convexity of the cost functions. We establish the existence of a Nash equilibrium by constructing a feasible strategy profile for all users, such that no user wishes to unilaterally deviate from its routing strategy. We first provide the following definition.

Definition 1. *A profile,* $\mathbf{f}$*, is referred to as balanced, if for any two paths* $p_1, p_2 \in \mathcal{P}$ *with* $f_{p_1} > 0$*, it holds that,* $b_{p_1}(\mathbf{f}) \leq \max_{e \in p_2}\{T_e(f_e)\}$.

Thus, at a *balanced* flow profile, for any two paths $p_1, p_2 \in \mathcal{P}$ with positive flow, their bottlenecks are equal, $b_{p_1}(\mathbf{f}) = b_{p_2}(\mathbf{f})$.

Definition 2. *A profile,* $\mathbf{f}$*, is referred to as proportional, if for any path* $p \in \mathcal{P}$*, and for any user* $i \in \mathcal{N}$*,* $f_p^i = \frac{r^i}{R} f_p$.

To demonstrate that a proportional profile is feasible, it needs to satisfy **(i)** the non-negativity constraint and **(ii)** the demand constraint of all users. Consider a user $i \in \mathcal{N}$. It follows that $f_p^i = \frac{r^i}{R} f_p \geq 0$, thus the non-negativity constraint is satisfied. Furthermore, $\sum_{p \in \mathcal{P}} f_p^i = \frac{r^i}{R} \sum_{p \in \mathcal{P}} f_p = r^i$, thus the demand constraint is also satisfied. In order to construct a feasible Nash equilibrium, we first establish following lemma.

Lemma 1. *Consider a bottleneck routing game. Any system optimal strategy profile is balanced.*

Proof. See Appendix A.1. □

We continue to construct a feasible profile, which is also a Nash equilibrium. Specifically, we focus on a profile that is proportional and system optimal.

Theorem 1. *Consider a bottleneck routing game, where for any user* i*,* $\alpha^i \in [1/2, 1]$. *Each system optimal proportional profile is a Nash equilibrium.*

Proof. Consider a system optimal, proportional profile, $\mathbf{f}$. As a result of Lemma 1, $\mathbf{f}$ is balanced, thus for all $i \in \mathcal{N}$, $b^i(\mathbf{f}) = b(\mathbf{f})$. Therefore, for any user i,

$$J^i(\mathbf{f}) = \alpha^i b^i(\mathbf{f}) - (1 - \alpha^i) \min_{j \neq i}\{b^j(\mathbf{f})\} = (2\alpha^i - 1) \cdot b(\mathbf{f}). \tag{6}$$

Assume by contradiction that $\mathbf{f}$ is not a Nash equilibrium. In other words, there exists a user i, which can send its flow according to $\bar{\mathbf{f}}^{\mathbf{i}} \neq \mathbf{f}^{\mathbf{i}}$ and by doing so, decreases its cost. Moreover, consider the case that the cost of the bottleneck link of i's rival, has increased due to i's deviation, i.e.,

$$\min_{j \neq i}\{b^j(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{-\mathbf{i}})\} > \min_{j \neq i}\{b^j(\mathbf{f})\}. \tag{7}$$

Denote the bottleneck link of i's rival at $(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{-\mathbf{i}})$ as n, thus

$$\min_{j \neq i}\{b^j(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{-\mathbf{i}})\} \equiv T_n(\bar{f}_n^i + f_n^{-i}). \tag{8}$$

Since $\mathbf{f}$ is balanced, from (8) it follows that,

$$T_n(\bar{f}_n^i + f_n^{-i}) > \min_{j \neq i}\{b^j(\mathbf{f})\} = b(\mathbf{f}) \geq T_n(f_n). \tag{9}$$

From (9) and Assumption A2 it follows that $\bar{f}_n^i > 0$. Therefore from (6), (7) and (9),

$$\begin{aligned} J^i(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{-\mathbf{i}}) &= \alpha^i b^i(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{-\mathbf{i}}) - (1 - \alpha^i) \min_{j \neq i}\{b^j(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{-\mathbf{i}})\} \\ &= \alpha^i b^i(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{-\mathbf{i}}) - (1 - \alpha^i) T_n(\bar{f}_n^i + f_n^{-i}) \\ &\geq (2\alpha^i - 1) T_n(\bar{f}_n^i + f_n^{-i}) \geq (2\alpha^i - 1) b(\mathbf{f}) = J^i(\mathbf{f}). \end{aligned}$$

The last inequality follows from (9) and from $\alpha^i \in [1/2, 1]$. Therefore, $J^i(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{-\mathbf{i}}) \geq J^i(\mathbf{f})$, which is a contradiction. We now consider the case where $\mathbf{f}$ is not a Nash equilibrium and

$$\min_{j \neq i}\{b^j(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{-\mathbf{i}})\} \leq \min_{j \neq i}\{b^j(\mathbf{f})\}. \tag{10}$$

Since $\mathbf{f}$ is system optimal, it holds that

$$b(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{-\mathbf{i}}) \geq b(\mathbf{f}). \tag{11}$$

Denote the bottleneck link of the system, at $(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{-\mathbf{i}})$ as s and consider the case where $\bar{f}_s^i = 0$. By definition $f_s^{-i} > 0$, otherwise s cannot be the system's bottleneck. Since, $f_s^{-i} > 0$ it follows that $f_s > 0$ and $f_s^i = \frac{r^i}{R} f_s > 0$. Consequently, from (11) and Assumption A2, $T_s(f_s^i + f_s^{-i}) > T_s(\bar{f}_s^i + f_s^{-i}) \geq b(\mathbf{f})$, which is a contradiction to s being the system's bottleneck. Therefore, $\bar{f}_s^i > 0$ and

$$b(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{-\mathbf{i}}) \equiv T_s(\bar{f}_s^i + f_s^{-i}) = b^i(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{-\mathbf{i}}). \tag{12}$$

Finally, from (6), (10), (11) and (12)

$$\begin{aligned}
J^i(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{-\mathbf{i}}) &= \alpha^i b^i(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{-\mathbf{i}}) - (1-\alpha^i)\min_{j \neq i}\{b^j(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{-\mathbf{i}})\} \\
&\geq \alpha^i b^i(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{-\mathbf{i}}) - (1-\alpha^i)\min_{j \neq i}\{b^j(\mathbf{f})\} \\
&= \alpha^i b(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{-\mathbf{i}}) - (1-\alpha^i)\min_{j \neq i}\{b^j(\mathbf{f})\} \\
&\geq \alpha^i b(\mathbf{f}) - (1-\alpha^i)\min_{j \neq i}\{b^j(\mathbf{f})\} = (2\alpha^i - 1) \cdot b(\mathbf{f}) = J^i(\mathbf{f}),
\end{aligned}$$

which is a contradiction. Thus, any system optimal proportional balanced flow is a Nash equilibrium. □

Theorem 1 illustrates that in any bottleneck routing game where for each user i, $\alpha^i \in [1/2, 1]$, there exists a Nash equilibrium. Moreover, there always exists a Nash equilibrium, which is system optimal[1]. This brings us to the following conclusion.

Corollary 1. *Consider a bottleneck routing game, where for any user i, $\alpha^i \in [1/2, 1]$. The Price of Stability is equal to 1.*

Even though Theorem 1 establishes the existence of desirable equilibria from a system's perspective, it might also happen that the selfishness of the users degrades the system substantially. This deficiency is captured by the Price of Anarchy.

Theorem 2. *Consider a bottleneck routing game, where for any user i, $\alpha^i \in [1/2, 1]$. The Price of Anarchy is unbounded.*

Proof. We establish the theorem through the following example.

[1] In [5] a similar theorem was proven for a more general topology. However, they only considered selfish users (i.e, $\forall i,\ \alpha^i = 1$).

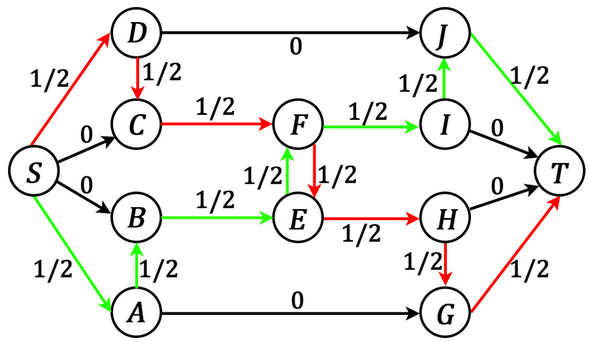

Fig. 2. Example of a network with an unbounded PoA.

Example 1. Consider the network $G = (V, E)$ as depicted in Figure 2. Further, consider two users i and j, each with a flow demand of $r^i = r^j = \frac{R}{2}$ and $\alpha^i = \alpha^j \equiv \alpha \in [1/2, 1]$. For any edge $e \in E$, the cost per unit of flow is equal to $T_e(f_e) = e^{f_e} - 1$. We focus on a specific profile $\mathbf{f}$, in which user i sends its total demand on a single path, namely $\{S, A, B, E, F, I, J, T\}$, and, user j sends its demand on the path $\{S, D, C, F, E, H, G, T\}$. The labels on the edges in Figure 2 correspond to the portion of the total flow that transverses on that edge at $\mathbf{f}$, i.e., f_e/R. Thus,

$$J^i(\mathbf{f}) = \alpha b^i(\mathbf{f}) - (1-\alpha)b^j(\mathbf{f}) = (2\alpha - 1) \cdot (e^{R/2} - 1).$$

It is straightforward that $J^i(\mathbf{f}) = J^j(\mathbf{f})$. Now assume by contradiction that $\mathbf{f}$ is not a Nash equilibrium. Hence, there exists a different routing strategy for user i, $\bar{\mathbf{f}}^{\mathbf{i}} \neq \mathbf{f}^{\mathbf{i}}$ at which user i can decrease its cost. If i places a positive flow on either (S, D), (S, C) or (A, G), it is immediate that $b^i(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{\mathbf{j}}) = b^j(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{\mathbf{j}}) > b^i(\mathbf{f})$ and $J^i(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{\mathbf{j}}) > J^i(\mathbf{f})$.

Thus, if i wishes to refrain from increasing its cost, it will send all its flow on (B, E) and its bottleneck will be at least $T_{(B,E)}(\frac{R}{2}) = e^{R/2} - 1$. It follows that at $(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{\mathbf{j}})$, there cannot exist an edge on which both i and j send a positive amount of flow, otherwise i increases its cost. Thus, the bottleneck of j stays the same. Hence,

$$\begin{aligned} J^i(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{\mathbf{j}}) = \alpha b^i(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{\mathbf{j}}) - (1-\alpha) b^j(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{\mathbf{j}}) &= \alpha b^i(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{\mathbf{j}}) - (1-\alpha) b^j(\mathbf{f}) \\ &\geq (2\alpha - 1) \cdot (e^{R/2} - 1) = J^i(\mathbf{f}), \end{aligned}$$

which is a contradiction. Because the users i and j are symmetric, the above analysis also holds for j. Therefore $\mathbf{f}$ is a Nash equilibrium. The bottleneck of the system at $\mathbf{f}$ is equal to $b(\mathbf{f}) = e^{R/2} - 1$.

On the other hand, at the system optimum, $\mathbf{f}^*$, an amount of flow, $R/4$, is sent through the following four paths: **1:** $\{S, A, G, T\}$, **2:** $\{S, B, E, H, T\}$, **3:** $\{S, C, F, I, T\}$, **4:** $\{S, D, J, T\}$. The system bottleneck at $\mathbf{f}^*$ is equal to $b(\mathbf{f}^*) = e^{R/4} - 1$. As a result, the Price of Anarchy in our example is lower bounded by

$$PoA = \frac{e^{R/2} - 1}{e^{R/4} - 1} \geq \frac{e^{R/2} - 1}{e^{R/4}} = e^{R/4} - \frac{1}{e^{R/4}} \geq e^{R/4} - 1.$$

Since R can be any positive number, the PoA is unbounded. □

4 Additive Routing Games

In this section we consider additive performance measures, such as delay, jitter and packet loss. Similar to bottleneck routing games, we first need to prove the existence of a Nash equilibrium. As mentioned in Section 3, due to the lack of quasi-convexity we cannot rely on well-known existence proofs for convex-games, such as the one given in [21]. We establish the existence of a Nash equilibrium by constructing a feasible strategy profile for all users, such that no user wishes to unilaterally deviate from its routing strategy. Moreover, we consider the specific case where for all $i \in \mathcal{N}$, $\alpha^i = 1/2$.[2] In other words, each user views its own performance and that of its rival, with equal importance. From (3), the cost of user i turns into

$$J^i(\mathbf{f}) = \sum_{l \in \mathcal{L}} \frac{f_l^i}{r^i} T_l(f_l) - \min_{j \neq i} \left\{ \sum_{l \in \mathcal{L}} \frac{f_l^j}{r^j} T_l(f_l) \right\}. \tag{13}$$

Note that we disregard $\alpha_i \equiv \alpha = 1/2$ from our equilibrium analysis, since it multiplies all users' costs by the same constant. We now bring the following definition from [27].

Definition 3. *A profile,* $\mathbf{f}$*, is a Wardrop equilibrium if for any two links* $l, n \in \mathcal{L}$ *with* $f_l > 0$*,* $T_l(f_l) \leq T_n(f_n)$*.*

In any additive routing game, there exists a Wardrop equilibrium. Moreover, it is unique with respect to the aggregated link flows f_l, [13, 27]. We focus on a specific Wardrop equilibrium, which is also proportional in the sense of Definition 2. Hence, it is also unique with respect to the individual user flows.

Theorem 3. *Consider an additive routing game as described in Section 2, where for all users* i*,* $\alpha^i = 1/2$*. There exists a Nash equilibrium. In particular, it is equal to the proportional Wardrop equilibrium.*

Proof. We consider the unique proportional Wardrop equilibrium, $\mathbf{f}$, and prove that no user wishes to unilaterally deviate from $\mathbf{f}$. Assume by contradiction that $\mathbf{f}$ is not a Nash equilibrium. Hence, there exists a user i and a routing strategy , $\bar{\mathbf{f}}^{\mathbf{i}} \neq \mathbf{f}^{\mathbf{i}}$ such that $J^i(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{-\mathbf{i}}) < J^i(\mathbf{f})$. We split the set of links $\mathcal{L}$, into three subsets: $\mathcal{L}^+ = \{l \in \mathcal{L} | \bar{f}_l^i > f_l^i\}$, $\mathcal{L}^- = \{l \in \mathcal{L} | \bar{f}_l^i < f_l^i\}$ and $\mathcal{L}^0 = \{l \in \mathcal{L} | \bar{f}_l^i = f_l^i\}$. Since $\bar{\mathbf{f}}^{\mathbf{i}} \neq \mathbf{f}^{\mathbf{i}}$, it follows that $\mathcal{L}^+$ and $\mathcal{L}^-$ are not empty. For any link $l \in \mathcal{L}^+$, denote $\epsilon_l \equiv \bar{f}_l^i - f_l^i$ and for any link $l \in \mathcal{L}^-$, denote $\delta_l \equiv f_l^i - \bar{f}_l^i$. Since r^i is constant, the differences in $\mathcal{L}^+$ and $\mathcal{L}^-$ are equal and $\sum_{l \in \mathcal{L}^+} \epsilon_l = \sum_{l \in \mathcal{L}^-} \delta_l$.

Because $\mathbf{f}$ is a proportional profile, it holds that for any two users $i, k \in \mathcal{N}$ and for any link $l \in \mathcal{L}$, $f_l^i / r^i = f_l^k / r^k$. Thus, for any link $l \in \mathcal{L}^0$ and any user $k \in \mathcal{N}$,

$$\left[\frac{\bar{f}_l^i}{r^i} - \frac{f_l^k}{r^k} \right] = \left[\frac{f_l^i}{r^i} - \frac{f_l^k}{r^k} \right] = 0. \tag{14}$$

[2] An existence and uniqueness proof for selfish users is given in [20].

Equation (14) holds for any $k \in \mathcal{N}$, hence also for i's rival at $(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{-\mathbf{i}})$. Denote i's rival at $(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{-\mathbf{i}})$ as j. Combining (14) with (13), we get

$$\begin{aligned}
J^i(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{-\mathbf{i}}) &= \sum_{l\in\mathcal{L}^+} \left[\frac{\bar{f}_l^i}{r^i} - \frac{f_l^j}{r^j}\right] T_l(\bar{f}_l^i + f_l^{-i}) + \sum_{l\in\mathcal{L}^-} \left[\frac{\bar{f}_l^i}{r^i} - \frac{f_l^j}{r^j}\right] T_l(\bar{f}_l^i + f_l^{-i}) \\
&\quad + \sum_{l\in\mathcal{L}^0} \left[\frac{\bar{f}_l^i}{r^i} - \frac{f_l^j}{r^j}\right] T_l(\bar{f}_l^i + f_l^{-i}) \qquad (15) \\
&= \sum_{l\in\mathcal{L}^+} \left[\frac{f_l^i + \epsilon_l}{r^i} - \frac{f_l^j}{r^j}\right] T_l(f_l + \epsilon_l) + \sum_{l\in\mathcal{L}^-} \left[\frac{f_l^i - \delta_l}{r^i} - \frac{f_l^j}{r^j}\right] T_l(f_l - \delta_l) \\
&= \sum_{l\in\mathcal{L}^+} \frac{\epsilon_l}{r^i} T_l(f_l + \epsilon_l) - \sum_{l\in\mathcal{L}^-} \frac{\delta_l}{r^i} T_l(f_l - \delta_l) > \sum_{l\in\mathcal{L}^+} \frac{\epsilon_l}{r^i} T_l(f_l) - \sum_{l\in\mathcal{L}^-} \frac{\delta_l}{r^i} T_l(f_l).
\end{aligned}$$

The last inequality follows from Assumption A2. Since $\mathbf{f}$ is a Wardrop equilibrium, we make two observations, namely

(1): $\forall l \in \mathcal{L}^-$, $\delta_l > 0$, thus $f_l > 0$. Therefore, from Definition 3, for any two links $l, n \in \mathcal{L}^-$, $T_l(f_l) = T_n(f_n)$.

(2): From Definition 3 it follows that for any link $l \in \mathcal{L}^+$ and any link $n \in \mathcal{L}^-$, $T_l(f_l) \geq T_n(f_n)$.

Consider a link $e \in \mathcal{L}^-$. Consequently, equation (15) turns into

$$J^i(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{-\mathbf{i}}) > \sum_{l\in\mathcal{L}^+} \frac{\epsilon_l}{r^i} T_e(f_e) - \sum_{l\in\mathcal{L}^-} \frac{\delta_l}{r^i} T_e(f_e) = \frac{1}{r^i} T_e(f_e) \cdot \left[\sum_{l\in\mathcal{L}^+} \epsilon_l - \sum_{l\in\mathcal{L}^-} \delta_l\right] = 0.$$

On the other hand, because $\mathbf{f}$ is proportional, it follows from (13) that

$$J^i(\mathbf{f}) = \sum_{l\in\mathcal{L}} \frac{f_l^i}{r^i} T_l(f_l) - \min_{k\neq i}\{\sum_{l\in\mathcal{L}} \frac{f_l^k}{r^k} T_l(f_l)\} = \sum_{l\in\mathcal{L}} \frac{f_l^i}{r^i} T_l(f_l) - \sum_{l\in\mathcal{L}} \frac{f_l^i}{r^i} T_l(f_l) = 0.$$

Thus, $J^i(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{-\mathbf{i}}) > J^i(\mathbf{f})$, which is a contradiction. Hence, $\mathbf{f}$ is a Nash equilibrium.

Now that we have proven the existence of a Nash equilibrium, we continue to investigate its uniqueness. We focus on a special case in which the network has two users, i.e., $N = 2$, and we denote these two users as i and j. It follows from (13) that $J^j(\mathbf{f}) = -J^i(\mathbf{f})$. In order to prove the Nash equilibrium's uniqueness, we use of the following lemma.

Lemma 2. *Consider an additive routing game as described in Section 2, where $N = 2$ and $\alpha^i = \alpha^j = 1/2$. At any Nash equilibrium $\mathbf{f}$, $J^i(\mathbf{f}) = J^j(\mathbf{f}) = 0$.*

Proof. See Appendix A.2.

We are now ready to prove our theorem.

Theorem 4. *For $N = 2$, the proportional Wardrop equilibrium is the unique Nash equilibrium, i.e., the Nash equilibrium is unique in the users' individual flows.*

Proof. See Appendix A.3.

An immediate consequence of Theorem 4 is that, the PoA of two-user systems is bounded by well-known bounds on the Wardrop equilibrium, e.g., see [19, 25].

Although Theorem 4 holds for a network with two users, in the general case of N-players, it does not hold. Indeed, in Appendix A.4 we provide an example of a network with N users and multiple Nash equilibria. Finally, in Appendix A.5, we provide an example of a network with users for which $\alpha^i \in [0, 1]$, in which no Nash equilibrium exists for either bottleneck routing games or additive routing games.

5 Conclusions

In this study we investigated routing games where the cost of each agent is represented as a combination of its own performance and that of its *rival*. We established the existence of Nash equilibria in games with bottleneck performance measures and games with additive performance measures. For bottleneck routing games and agents with $\alpha^i \in [1/2, 1]$, namely, games where agents care more about their own performance than that of their rivals', we established that the Price of Stability is equal to 1, i.e., a system optimal Nash equilibrium always exists. Moreover, we provide an example in which the Price of Anarchy is unbounded. For additive routing games, we focused on the fundamental load balancing game of routing over parallel links and on agents with $\alpha^i = 1/2$, namely, games where agents view their own performance and that of their rivals with equal importance. We proved that the proportional Wardrop equilibrium (which exists and is unique) is also a Nash equilibrium. Moreover, for a two-player system, we established the uniqueness of the Nash equilibrium. In this case, the PoA can be bounded by well-known bounds on the Wardrop equilibrium. We also provided an example of the non-uniqueness of the Nash equilibrium for a system with N-players, and an example of its non-existence for agents with $\alpha^i \in [0, 1]$. In future research, it would be interesting to consider networks with multiple sources and destination pairs. Lastly, establishing the existence of a Nash equilibrium for additive games and agents with $\alpha^i \in [1/2, 1]$, remains an open problem.

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A Appendix

A.1 Proof of Lemma 1

Proof. Consider a system optimal profile $\mathbf{f}^*$. Let the set $\mathcal{P}_{\mathbf{f}^*}$ contain all paths with positive flow, whose bottlenecks are equal to the bottleneck of the system. Thus, for any path $p \in \mathcal{P}_{\mathbf{f}^*}$,

$$b_p(\mathbf{f}^*) = b(\mathbf{f}^*).$$

Now assume by contradiction that the system optimal strategy profile $\mathbf{f}^*$ is not balanced. Therefore, the set $\mathcal{P} \backslash \mathcal{P}_{\mathbf{f}^*}$ is not empty, and for all $q \in \mathcal{P} \backslash \mathcal{P}_{\mathbf{f}^*}$ and all $p \in \mathcal{P}_{\mathbf{f}^*}$,

$$b_p(\mathbf{f}^*) > b_q(\mathbf{f}^*). \tag{16}$$

Consider a path $q \in \mathcal{P} \backslash \mathcal{P}_{\mathbf{f}^*}$. We construct a different feasible routing strategy $\mathbf{f}$ by sending a small amount of flow, $\epsilon > 0$ from all the paths in $\mathcal{P}_{\mathbf{f}^*}$ to path q. Specifically, for all $p \in \mathcal{P}_{\mathbf{f}^*}$, $f_p = f_p^* - \epsilon$, $f_q = f_q^* + |\bar{\mathcal{P}}| \cdot \epsilon$, for some small $\epsilon > 0$, and for all other paths, $p \in \mathcal{P} \backslash \{\mathcal{P}_{\mathbf{f}^*} \ \cup \ q\}$, $f_p = f_p^*$. If we consider a small enough ϵ, it follows from the strict inequality of (16) that for all $q \in \mathcal{P} \backslash \mathcal{P}_{\mathbf{f}^*}$ and for all $p \in \mathcal{P}_{\mathbf{f}^*}$

$$b_p(\mathbf{f}^*) > b_p(\mathbf{f}) > b_q(\mathbf{f}) > b_q(\mathbf{f}^*). \tag{17}$$

In other words, for a small enough ϵ,

$$b(\mathbf{f}^*) > b(\mathbf{f}). \tag{18}$$

By constructing a new routing strategy $\mathbf{f}$ we are able to lower the bottleneck of the system, which is a contradiction to the optimality of $\mathbf{f}^*$.

A.2 Proof of Lemma 2

Proof. Assume by contradiction that $J^i(\mathbf{f}) > 0$. Consider a different routing strategy $\bar{\mathbf{f}}^{\mathbf{i}}$ in which for any link $l \in \mathcal{L}$, i sends its flow according

$$\bar{f}_l^i = \frac{r^i}{r^j} f_l^j.$$

$\bar{\mathbf{f}}^{\mathbf{i}}$ is a feasible routing strategy for i, since **(1)** $\forall l \in \mathcal{L},\ \bar{f}_l^i \geq 0$ and **(2)** $\sum_{l \in \mathcal{L}} \bar{f}_l^i = \sum_{l \in \mathcal{L}} \frac{r^i}{r^j} f_l^j = r^i$. Moreover, from (13) it follows that $J^i(\bar{\mathbf{f}}) = 0$, which contradicts the fact that $\mathbf{f}$ is a Nash equilibrium. Thus, at any Nash equilibrium $\mathbf{f}$, $J^k(\mathbf{f}) \leq 0$, for $k = i, j$.

Now assume by contradiction that $J^i(\mathbf{f}) < 0$. Thus, $J^j(\mathbf{f}) > 0$, which is a contradiction. □

A.3 Proof of Theorem 4

Proof. Denote the proportional Wardrop equilibrium as $\hat{\mathbf{f}}$ and assume by contradiction that there exists another Nash equilibrium $\mathbf{f} \neq \hat{\mathbf{f}}$. From Lemma 2 it follows that $J^i(\mathbf{f}) = J^j(\mathbf{f}) = 0$. Consider a new routing strategy for user i, $\bar{\mathbf{f}}^{\mathbf{i}} \neq \mathbf{f}^{\mathbf{i}}$. At $\bar{\mathbf{f}}^{\mathbf{i}}$, user i sends its flow according $\bar{f}_l^i = \frac{r^i}{r^j} f_l^j$ for any $l \in \mathcal{L}$. In Lemma 2, $\bar{\mathbf{f}}^{\mathbf{i}}$ is shown to be feasible. Furthermore, from (13) and Lemma 2, it follows that $J^i(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{\mathbf{j}}) = J^i(\mathbf{f}^{\mathbf{i}}, \mathbf{f}^{\mathbf{j}}) = 0$, hence i does not increase its cost by changing its strategy to $\bar{\mathbf{f}}^{\mathbf{i}}$. At the new routing strategy profile ($\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{\mathbf{j}}$), either one of the two cases holds:

Case 1: $(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{\mathbf{j}})$ is not a Wardrop equilibrium.
Case 2: $(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{\mathbf{j}})$ is a Wardrop equilibrium.

Consider Case 1. There must exist two links, $l, n \in \mathcal{L}$ such that $f_l > 0$ and $T_l(f_l) > T_n(f_n)$. We construct a new strategy for i, at which i sends a small amount of flow, $\epsilon > 0$, from link l to n. Denote this new strategy as $\tilde{\mathbf{f}}^{\mathbf{i}}$. At $(\tilde{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{\mathbf{j}})$, i's cost equals:

$$\begin{aligned} J^i(\tilde{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{\mathbf{j}}) = & \left[\frac{\bar{f}^i_l - \epsilon}{r^i} - \frac{f^j_l}{r^j} \right] T_l(f_l - \epsilon) \\ & + \left[\frac{\bar{f}^i_n + \epsilon}{r^i} - \frac{f^j_n}{r^j} \right] T_n(f_n + \epsilon) \\ & + \sum_{e \in \mathcal{L} \setminus \{l,n\}} \left[\frac{\bar{f}^i_e}{r^i} - \frac{f^j_e}{r^j} \right] T_e(f_e) \end{aligned} \tag{19}$$

Thus, (19) turns into:

$$J^i(\tilde{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{\mathbf{j}}) = \frac{\epsilon}{r^i} \left[T_n(f_n + \epsilon) - T_l(f_l - \epsilon) \right], \tag{20}$$

which is negative for a small enough ϵ. Hence, we constructed a feasible strategy in which user i decreases its cost, which is a contradiction to $\mathbf{f}$ being a Nash equilibrium.

Now consider Case 2. It follows that $(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{\mathbf{j}})$ is the unique proportional Wardrop equilibrium. However, in Theorem 3 it is proven that any unilateral deviation of user i from the proportional Wardrop equilibrium causes a strict increase in cost to user i. In other words,

$$J^i(\hat{\mathbf{f}}^{\mathbf{i}}, \hat{\mathbf{f}}^{\mathbf{j}}) = J^i(\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{\mathbf{j}}) < J^i(\mathbf{f}^{\mathbf{i}}, \mathbf{f}^{\mathbf{j}}),$$

which is a contradiction.

A.4 Example with multiple Nash equilibria in additive routing games.

Example 2. Consider a network with four users i, j, k, h and with two parallel links, for which $T_1(f_1) = f_1,\ T_2(f_2) = f_2$. The demand of each individual user is equal to 1. Consider the profile, $\mathbf{f}$, at which $f^i_2 = f^j_2 = f^k_1 = f^h_1 = 1$ and $f^i_1 = f^j_1 = f^k_2 = f^h_2 = 0$. It is clear that $H^z(\mathbf{f}) = 2$ for $z = i, j, k, h$. Thus, $J^z(\mathbf{f}) = 2 - 2 = 0$ for $z = i, j, k, h$.

Now consider user i and a different profile $\bar{\mathbf{f}} = (\bar{\mathbf{f}}^{\mathbf{i}}, \mathbf{f}^{-\mathbf{i}})$, in which i sends an amount of $0 < \epsilon \leq 1$ to link 1. It follows that $H^i(\bar{\mathbf{f}}) = \epsilon(2+\epsilon) + (1-\epsilon)(2-\epsilon)$, $H^j(\bar{\mathbf{f}}) = 2 - \epsilon$ and $H^k(\bar{\mathbf{f}}) = H^h(\bar{\mathbf{f}}) = 2 + \epsilon$. Thus,

$$J^i(\bar{\mathbf{f}}) = H^i(\bar{\mathbf{f}}) - H^j(\bar{\mathbf{f}}) = 2\epsilon^2 > 0.$$

Thus i increases its cost at $\bar{\mathbf{f}}^{\mathbf{i}}$. Since the example is symmetric, this holds for all users, which proves that $\mathbf{f}$ is a Nash equilibrium. However, $\mathbf{f}$ is not a proportional profile, thus according to Theorem 3, there must exist another Nash equilibrium.

A.5 Example of a network without a Nash equilibrium, for users with $\boldsymbol{\alpha^i \in [0,1]}$

Consider a network with two parallel links and two users with demands $r^1 = 1$, $r^2 = 1/2$. Moreover, $\alpha^1 = 1$ and $\alpha^2 = 0$. Consider the cost per unit of flow on the links, $T_1(f_1) = \frac{1}{1-f_1}$, $T_2(f_2) = \frac{1}{1-f_2}$.

User 2 aims to minimize its cost $J^2(\mathbf{f}) = -H^1(\mathbf{f})$. By sending its entire demand, r^2, on the link with $\max\{f_1^1, f_2^1\}$, user 2 is able to minimize its cost to $-\infty$. However, $J^1(\mathbf{f}) = \infty$, thus user 1 will respond by sending part of its flow on the other link. User 2 will then follow user 1 and again bring its own cost to $-\infty$. It is clear that for both bottleneck routing games and additive routing games, there does not exist any Nash equilibrium.