ON CONTINUOUS NETWORK FLOWS

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ABSTRACT

This work addresses two problems concerning continuous dynamic flows. A model is presented for a network that incorporates continuous time-varying flows, link capacities, node storage capacities, as well as time dependent link delays. It is an enhancement of previous results which do not incorporate time varying link delays. We present a generalized min-cut max-flow theorem for that model. A second result deals with universal flows, originally dealt with for the discrete case. We show how such flows can be constructed in a way that involves parallelism.

Keywords: Delays, flows, maximal flows, networks, time-dependency, universal flows.

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I. INTRODUCTION

The problem of finding a maximal flow in a given time interval between two nodes ("source" and "destination") of a network was first dealt with in the seminal work of Ford and Fulkerson[1]. Due to the importance of the problem a large variety of extensions in several directions followed (see, for example, Gondran and Minoux[2]). Most of these extensions fall into the category of "static" maximal flow. The problem of dynamic maximal flow was also treated by Ford and Fulkerson [3] but generated far less sequel work, mainly due to the higher complexity of the problem.

The work presented here falls into the category of dynamic flows. Aside from the original paper cited above several others are relevant to the work reported here. In his work Gale[4] proved the existence of a universal maximal flow, i.e., a flow that is maximal with respect to each period in the domain. Wilkinson[5], in a sequel work, presented an algorithm to find the universal maximal flow. Halpern[6], taking a different direction, presented an extension to the case with discretely time varying link capacities and node capacities.

All the above works dealt with a discrete version of time. Anderson et. al.[7] presented an extension that deals with the continuous case, namely link flows and capacities that are measurable functions of time, and node capacities that are continuous functions; however, their analysis is restricted to zero link delays (although an extension dealing with constant delays was discussed). In that work, which is the basis of the extension presented here, the authors define a "generalized cut", the continuous counterpart of the regular cut, and assign it values so that a min-cut max-flow theorem can be stated. However, the results of [7] are erroneous when applied to general measurable flow and capacity functions and were corrected in [8]. That later work also expands the results to apply to networks with constant (nonzero) link delays. A second result presented in [8] is the proof of an equivalence between three continuous flow problems: a version of a universal maximal flow problem, an evacuation problem, and a minimum-cost flow problem. However, no solution to any of these problems is presented.

The aim of this work is to enhance the previous results in two directions, namely: to deal with continuously varying link delays, and to give a solution to the universal maximal flow problem for the continuous case.

The introduction of time-dependent link delays seems necessary in an environment where all other parameters change with time. Time-dependent delays were treated in the context of shortest-path problems (e.g., [9,10,11]), where it was shown that such delays introduce nontrivial complications to the solution. Those results hint that time-dependent delays should be treated with care also when dealing with the maximal flow problem. Indeed, we shall see that some properties of such delays demand different definitions of cuts and reachable nodes (although they do keep a basic resemblance to the original definition). The most remarkable of these properties is the fact that flow incoming at a node may concentrate at certain points in time to create *singular* jumps in the amount of material stored at that node. In fact, one contribution of this paper is the investigation of flows that are "smooth" on the sender's side but contain such singularities at the receiver's side.

Universal maximal flows are clearly of interest, because of their outstanding property (being maximal for all moments within an interval), and also due to the relevance shown in [8] of such flows to evacuation problems. The latter are fundamental problems in several network applications, e.g.: optimal dynamic routing [12,13,14].

As previously mentioned, the results of [7] cannot be applied to general measurable flows. These results do hold for "well behaved" functions that are at least piecewise-continuous. The results of [8] indeed apply to general measurable functions, but at the expense of complicating the model and the basic results (general cuts, reachability etc). The definitions of [8] while encompassing a wider scope are more complicated and result in a more complicated algorithm which, in our opinion, comes at the expense of a more insightful look at the nature of the problem. Since this paper is intended to show the particular nature of the case involving time-dependent delays, we prefer to base our work on the simpler model, thereby retaining the intuitive and much simpler extension of static cuts to the continuous case. It is pointed out that, in any reasonable application, the class of piecewise-continuous functions is sufficiently broad to describe the behavior of network functions. From a practical standpoint it is clear that an implementation of maximal flow algorithms that deals with functions that are not even piecewise-continuous would be intractable. It seems plausible to state that for sufficiently "well behaved" networks a maximal piecewise-continuous flow does exist. Till recently, this has been proven for only very restricted cases (e.g., piecewise-constant arc capacities and piecewise-linear nodal storage capacities)[15]. Recently, the existence of a maximum piecewise-continuous flow for networks with piecewise-analytic arc capacities and nodal storage capacities has been proven[16].

This paper is structured as follows. Section II presents the extended model that incorporates time dependent link delays. In Section III we formulate the maximal flow problem, define the concepts of generalized cuts and reachable nodes appropriate to this model, and present a (restricted) minimum-cut maximal-flow theorem. In Section IV we discuss a scheme for constructing universal maximal flows.

II. THE MODEL

The model presented here is a direct extension of the model presented in [7]. Let us define a network $\Omega=(\Lambda,E)$ where $\Lambda=\{1,2,\ldots,n\}$ is the set of nodes of which node 1 is considered the source and node n the sink of all flows. $E=\Lambda\times\Lambda$ is the set of links. To each link $(j,k)\in E$ we assign a nonnegative and piecewise-continuous function $b_{jk}(t)$ which denotes the capacity of that link. To each node j we assign a continuous nonnegative function $a_j(t)$ which denotes the storage capacity of that node. We denote by $x_{jk}(t)$ the total flow exiting node j at time t on link (j,k), and by $\overline{x}_{jk}(t)$ the total flow entering node k at time t on link (j,k) (the difference between the two will become clear soon). The actual storage at node j at time t is denoted $y_j(t)$. All functions are assumed to be defined on some finite interval $[0,T]$. A network flow is a set of piecewise-continuous link flows $\{x_{jk}(t)\}$, $(j,k)\in E$, which obey the delay rules. If a network flow obeys the capacity constraints, then we say that it is feasible. We shall use the term "flow" to refer to both link and network flows. For any given $t_0\leq T$ the volume of "material" flowing into node n over the interval $[0,t_0]$ is

$$v(t_0)=\int_0^{t_0}\sum_{k=1}^{n}[\overline{x}_{kn}(t)-x_{nk}(t)]\,dt$$

Our objective is to maximize this quantity in various circumstances.

A. Time Dependent link Delays

To extend the flow problem to time-dependent link delays we describe here briefly the link model and some of its properties. More detailed information can be found in [10].

We define for each link $(i,j)\in E$ a nonnegative continuous[1] and piecewise-differentiable function $d_{jk}(t)$ signifying link delays. A unit of flow leaving node j on link (j,k) at time t will arrive at node k at time $t+d_{jk}(t)$. This model is fairly general and includes, as a special case, the situation where delays change while the material is flowing through the link[10]. This model has the following properties:

i. To each interval I_d of departure times from node j there corresponds an interval I_a of arrival times at node k, i.e., I_a is the range of $t+d_{jk}(t)$ for domain I_d. Note that I_a may collapse to a single point.

ii. Links may not obey the 'FIFO rule' at all times i.e., for two departure times from node j $t_{d_1}<t_{d_2}$ the corresponding arrival times at node k may fulfill $t_{a_1}\geq t_{a_2}$.

[1]Our treatment can be extended to piecewise continuous functions as well; for brevity we shall focus the presentation on the continuous case.

iii. There may be an arrival time t_a at node *k* to which corresponds a whole interval I_d of departure times from node *j*. This happens if $\forall t \in I_d \ t + d_{jk}(t) = t_a$. Such arrival times will be referred to as *singular arrival times* for link (j,k).

Property (iii) above is a major difference between the constant delay and varying delay problems. Note that flow incoming to node *k* at a singular arrival time for link (j,k) may cause a discrete jump in the amount of material at that node. Our extension deals to a large extent with treating cases resulting from this property.

We introduce the following definitions regarding departure and arrival times within $[0,T]$. They apply to every link (j,k).

- The set of all departure times which cause an arrival time *t* to be singular is denoted $S_{jk}(t)$, i.e., $S_{jk}(t) \stackrel{\Delta}{=} \{\tau | \exists \ \delta > 0 \ \forall \tau' \quad \tau - \delta < \tau' < \tau + \delta, \ \tau' + d_{jk}(\tau') = t\}$. We shall refer to these times as the *singular* departure times. By its definition the set does not include the endpoints of the intervals.
- The singular departure times within the interval *I* for arrival time *t* are denoted $S_{jk}(I,t)$, i.e., $S_{jk}(I,t) \stackrel{\Delta}{=} I \bigcap S_{jk}(t)$.
- The nonsingular departure times for arrival time *t* are denoted $N_{jk}(t)$, i.e., $N_{jk}(t) \stackrel{\Delta}{=} \{\tau | \tau + d_{jk}(\tau) = t, \quad \tau \notin \ S_{jk}(t)\}$. It is straightforward that $N_{jk}(t)$ is a countable set.

To summarize these definitions consider figure 1 which depicts some link delay function $d_{jk}(t)$. $S_{jk}(t_2)$, the singular departure times at node *j* with respect to node *k* and time t_2 are shown. The circle marked points are $N_{jk}(t_2)$ and the x-marked points are $N_{jk}(t_1)$.

Figure 1: Definition of sets of departure and arrival times

III. THE MAXIMAL FLOW PROBLEM

A. Problem Formulation

The following are definitions regarding flows incoming to a node.

- $\hat{x}_{kj}(t)$ is the <u>nonsingular</u> flow coming into node j on link (k,j) at time t, that is $\hat{x}_{kj}(t) \overset{\Delta}{=} \sum_{t' \in N_{kj}(t)} x_{kj}(t')$. (Note that for all t: $\int_{N_{kj}(t)} x_{kj}(t')\,dt' = 0$.)
- $\tilde{x}_{kj}(t)$ is the <u>singular</u> flow coming into node j on link (k,j) at time t, that is $\tilde{x}_{kj}(t) \overset{\Delta}{=} \int_{S_{kj}(t)} x_{kj}(t')\,dt'$.
- $\overline{x}_{kj}(t)$ is the <u>total</u> flow coming into node j on link (k,j) at time t, that is $\overline{x}_{kj}(t) \overset{\Delta}{=} \hat{x}_{kj}(t) + \tilde{x}_{kj}(t)\,\delta(t)$ where $\delta(t)$ is Dirac's impulse function.

The Time-Dependent Continuous Network Flow Problem (TCNFP) can be stated as follows:

$$\text{Maximize: } v(T) = \sum_{k=1}^{n-1} \int_0^T [\overline{x}_{kn}(\tau) - x_{nk}(\tau)]\,d\tau$$

Subject to (for $0 \le t \le T$):

1. $\sum_{k \ne j} \int_0^t [\overline{x}_{kj}(\tau) - x_{jk}(\tau)]\,d\tau \le a_j(t) \quad j = 2, 3, \ldots, n-1.$
2. $\sum_{k \ne j} \int_0^t [x_{jk}(\tau) - \overline{x}_{kj}(\tau)]\,d\tau \le 0 \quad j = 2, 3, \ldots, n-1.$
3. $x_{jk}(t) \le b_{jk}(t)$.
4. $x_{jk}(t) \ge 0$.

Note that the link capacity constraint applies to the outgoing flow. Thus, capacity constraints are imposed on flows leaving a node, but nodes admit singular incoming flows. To illustrate such a case, consider a situation in which a (possibly very large) number of parallel links exist between two nodes of a real network, each with a finite capacity constraint. At each instant of time, only a small part of the links admits flow, hence in our model, in which all links are modeled by a single link, the capacity is also finite. However, due to the time-dependent behavior of link delays, large ("singular") quantities of flow can arrive at the same time to the second node, from all parallel links.

B. Coping with Time-Dependent Delays

In [7] the above problem was solved for zero delays. This was done through the following extension of the min-cut max-flow theorem.

A generalized cut in Ω separating nodes 1 and n was defined as "a set valued function of time C, defined on $[0,T]$, with $C(t)$ a subset of Λ such that for every $t \in [0,T]$, $1 \in C(t)$, $n \in \overline{C}(t)$ and for each j in Λ, the set $\Gamma_j = \{t | j \in C(t)\} \cap (0,T)$ is open". Thus, Γ_j could be represented as a countable union of disjoint open intervals, i.e.:

$$\Gamma_j = \bigcup_{i=1}^{\infty} (\alpha_{ij}, \beta_{ij})$$

For dealing with constant delays, the following definition was introduced:

$$\Gamma_{jk} \stackrel{\Delta}{=} \{t | t \in \Gamma_j, t + d_{jk} \notin \Gamma_k, t + d_{jk} \leq T\} \tag{1}$$

where d_{jk} are nonnegative constants representing link delays. The value V of the cut C was defined as

$$V = \sum_{j=1}^{n} \sum_{k=1}^{n} \int_{\Gamma_{jk}} b_{jk}(\tau)\, d\tau + \sum_{j=2}^{n-1} \sum_{t \in \{\beta_{ij}\} \backslash T} a_j(t) \tag{2}$$

when the second sum in the right hand-side of (2) exists, and $V = \infty$ otherwise.

It was proven in [7] (with the restriction presented in [15]) that the maximal value of $v(T)$ equals the minimal value of V. For a flow which is not maximal, a nonempty set of *reachable nodes* was defined, through which a feasible increase of flow can be achieved.

The generalized cuts defined in [7] must be modified to cope with the time dependency of links. To that end we must redefine Γ_{jk} and in light of equation (1) are inclined to use $\Gamma_{jk} \stackrel{\Delta}{=} \{t | t \in \Gamma_j, t + d_{jk}(t) \notin \Gamma_k, t + d_{jk}(t) \leq T\}$. However, such a modification is not sufficient. To demonstrate this consider figure 2. Suppose that $a_i(t) \equiv 0$ for all nodes. For $T = 1$ the maximal flow from node 1 to node 3 is obviously 0 since flow leaving node 1 in the interval [0,1] arrives at node 2 at $t = 1$. However all generalized cuts, as defined in [7] (with the above new definition of Γ_{jk}), have nonzero value. The problem stems from property (iii) of the time dependent link delays (see Section II) and the example hints at the

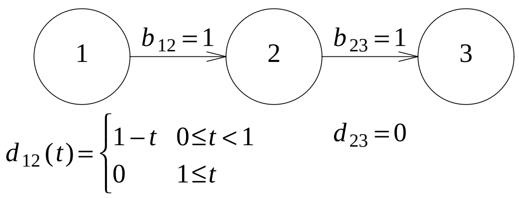

Figure 2: Cuts in a network with time-dependent link delays

solution. If the set Γ_j contained also discrete points (rather than only intervals) we could define $\Gamma_1 = \{ (0,1) \}$, $\Gamma_2 = \{ 1 \}$ and thus $C(t) = \{ 1 \}$ for $0 \le t < 1$ and $C(t) = \{ 1,2 \}$ for $t = 1$, and, indeed, this cut has a zero value.

Let us therefore define a generalized cut $C(t)$ in Ω as a set valued function of time defined on $[0,T]$ separating nodes 1 and n such that

1. For every t $1 \in C(t)$, $n \in \bar{C}(t)$.
2. For each $j \in \Lambda$ the set $\Gamma_j = \{ t \,|\, j \in C(t) \}$ can be expressed as a union of countable open intervals, unique up to order, and a countable set of discrete points, i.e., $\Gamma_j = \left[\bigcup_{i=1}^{\infty} \{ (\alpha_{ij}, \beta_{ij}) \} \right] \cup \left[\bigcup_{i=1}^{\infty} \{ \gamma_{ij} \} \right]$.
3. If α_{lj} is a singular arrival time for some link (k,j) then $\alpha_{lj} \in \{ \gamma_{ij} \}$.
4. If β_{lj} is a singular arrival time for some link (k,j) then $\beta_{lj} \in \{ \gamma_{ij} \}$.

Thus for each node j we can define the following sets:

$$R_j = [\bigcup_i \{ \beta_{ij}, \gamma_{ij} \} \setminus \{ \alpha_{ij} \}] \setminus \{ T \}$$

$$\Gamma_{jk} = \Gamma_j \bigcap \{ t \,|\, (t + d_{jk}(t)) \in \bar{\Gamma}_k, \quad t + d_{jk}(t) \le T \}$$

The value of the cut is defined as

$$V = \sum_{j=1}^{n} \sum_{k=1}^{n} \int_{\Gamma_{jk}} b_{jk}(t)\, dt + \sum_{j=2}^{n-1} \sum_{t \in R_j} a_j(t)$$

if the last summation is defined, and is assumed to be ∞ otherwise.

A second problem that arises when trying to apply the approach in [7] to the new model is concerned with the way by which a flow is augmented in order to reach its maximal value. Properties (i)-(iii) of the time dependent link delays require a more subtle treatment of augmentation intervals which calls for a redefinition of reachability as follows (in the following, reachable times are limited to $t \in [0,T]$):

i) Node 1 is reachable for every time $t \in [0,T]$.

ii) If node j is reachable at time t and there is an interval $I = (t - \delta, t + \delta)$ such that $\delta > 0$ and

$$\inf_{t' \in I} \{ b_{jk}(t') - x_{jk}(t') \} > 0,$$

then

ii.1) If $k = n$ or $S_{jk}(I, t + d_{jk}(t)) = \varnothing$ then k is reachable at time $t + d_{jk}(t)$.

ii.2) If $S_{jk}(I, t + d_{jk}(t)) \ne \varnothing$ then k is singularly reachable at time $t + d_{jk}(t)$.

iii)

iii.1) If node j is reachable at time t and there is an interval $I=(\hat{t}-\delta,\hat{t}+\delta)$ such that $\delta>0$, $\hat{t}+d_{kj}(\hat{t})=t$ and $\inf_{t'\in I}\{x_{kj}(t')\}>0$, and if $S_{kj}(I,t)=\emptyset$ then k is reachable at time $\hat{t}$.

iii.2) If node j is reachable or singularly reachable at time t and there is an interval $I=(\hat{t}-\delta,\hat{t}+\delta)$ such that $\delta>0$, $\hat{t}+d_{kj}(\hat{t})=t$ and $\inf_{t'\in I}\{x_{kj}(t')\}>0$, and if $S_{kj}(I,t)\neq\emptyset$ then k is reachable at time $\hat{t}$.

iv) If node j is reachable or singularly reachable at time t_1 and for every $t\in[t_1,t_2)$, $y_j(t)<a_j(t)$ then j is reachable at all times $t\in[t_1,t_2)$.

v) If node j is reachable or singularly reachable at time t_1 and for every $t\in(t_2,t_1]$, $y_j(t)>0$ then j is reachable at all times $t\in(t_2,t_1]$.

vi) If $y_j(T)>0$ ($j\neq n$), then node j is reachable at time T.

vii) If for $j=2,3,\ldots,n-1$ and for t such that $t+d_{jk}(t)>T$ there is an interval $I=(t-\delta,t+\delta)$ such that $\delta>0$ and $\inf_{t'\in I}\{x_{jk}(t')\}>0$, then node j is reachable at time t.

Reachable nodes are those having finite chains of nodes and arcs connecting them to the source through which a feasible increase in flow can be made. However, when rules (vi) or (vii) are invoked the increase in flow comes from residual storage at a node at time T, or from flow en route on links at that time, correspondingly. The definition of reachability ensures that the improvement in flow can be made on sets of nonzero measure.

The main differences between the above definition of reachability and that of [7] lie in the distinction made in rules (ii) and (iii) and in the use of *singular* reachability in rules (ii)-(v). Singular reachability represents the situation in which flow can get into the node but, because this flow arrives at a singular arrival time (rule (ii.2)) we must guarantee that either the node can store it for some time (rule (iv)) or else a singular decrease is possible, either of flow entering from some other node (rule (iii.2)) or of flow that has been stored in the node (rule (v)). The only exception to this is the destination node n, which is thus included in case (ii.1) of rule (ii). Note that a node j may be singularly reachable at some time τ without being (plainly) reachable. Such a node j can still participate in a flow augmentation path, by making a subsequent node be reachable through an application of rule (iii.2); this corresponds to replacing a singular flow arriving at j from one neighbor by a similar one from another neighbor.

Another difference is the addition of rule (vii), which should be introduced even if delays are constant but nonzero[8].

C. Min-Cut Max-Flow Theorem

We aim at deriving a theorem that, similarly to static flows, equates maximal flows with cuts of minimal value. The theorem is developed by three lemmas similar to those presented in [7]. The proofs are detailed in [17] and follow essentially along the lines of [7].

Lemma 1: Suppose that v is the value of any flow which is feasible for TCNFP and V is the value of any generalized cut C in Ω. Then it follows that $v \le V$. □

Lemma 2: Suppose for a feasible flow in the network Ω, that the sink is reachable at time $t \in [0,T]$. Then it is possible to construct a feasible flow in Ω with a value strictly greater than that of the existing flow. □

Lemma 3: Suppose that for a feasible flow in the network Ω that node n is not reachable for any $t \in [0,T]$. Define a generalized cut C by $C(t) = \{ j \mid j \textit{ is reachable at time } t \}$. If the value of the flow is v and if the value of C is V then $v = V$. □

The proof of the following min-cut max-flow theorem follows directly from lemmas 1-3.

Theorem 1: Suppose that $\{x_{jk}{}^{*}(t)\}$ is a (piecewise-continuous and feasible) maximal flow in the network Ω, i.e., its value v^{*} is such that $v^{*} \ge v$, where v is the value of any other flow; then v^{*} is equal to the minimum value V of all generalized cuts in Ω separating nodes 1 and n. □

D. Extensions

Certain simple expansions of our model and results are possible. For example, consider a network with capacity constraints on the *incoming* flow. Such constraints are easily incorporated in our model by imposing them "backwards" on the outgoing flow capacities (i.e., if $\hat{b}_{jk}(\tau)$, $\bar{b}_{jk}(\tau)$ are, correspondingly, the outgoing and incoming flow capacity functions for link (j,k), then we should assign $b_{jk}(\tau) = \min\{\hat{b}_{jk}(\tau), \bar{b}_{jk}(\tau + d_{jk}(\tau))\}$). In the same manner, we can impose capacity constraints on the flow passing through a link for every moment of traversal. Note that, with capacity constraints on the incoming flow, there are no singular flows. This means that the above extension is a particular case that is considerably simpler than the general case discussed in this paper.

IV. UNIVERSAL MAXIMAL FLOWS

Considering the general model and the TCNFP problem presented in the previous sections, a universal maximal flow (for the continuous case) is a collection of feasible flows $x_{jk}(t)$ on $[0,T]$ such that for each $t_0 \in [0,T]$ the value of

$$v(t_0) \overset{\Delta}{=} \int_0^{t_0} \sum_{k=1}^{n} [\overline{x}_{kn}(t) - x_{nk}(t)]\, dt$$

is maximized.

One might hope that when a maximal flow does exist, a universal maximal flow is guaranteed. Unfortunately, this is not the case [17]. We shall assume that a piecewise-continuous universal maximal flow exists. Our aim is to present a procedure for identifying such a flow, using the concepts of cuts and reachability. We point out that if flows are allowed to be general measurable functions, then the existence of a universal maximal flow for a network with constant link delays follows directly from Theorem 4 of [8].

A. Sufficient Conditions for Universality

In order to deal with universal flows, the definition of reachability is modified to be explicitly associated with some termination time. We say that node j is reachable at time t *relative to* termination time t_0 if node j is reachable at time t in the equivalent TCNFP defined on the same network with the interval $[0, t_0]$. Suppose that node k is reachable at time t (in the regular sense) by virtue of the fact that node j is reachable at time t' and by an application of one of the reachability rules (ii)-(v). It follows that if node j is reachable at time t' relative to t_0 and if $t \le t_0$ then node k is reachable at time t relative to time t_0. In order to initiate a chain of nodes which is reachable relative to some time, we introduce the following rules:

i´) For all $t \in [0, T]$, node 1 is reachable at time t relative to all times t_0, $t \le t_0 \le T$.

vi´) If $y_l(t_0) > 0$ ($l \ne n$) for $t_0 \in [0, T]$ then node l is reachable at t_0, relative to time t_0.

vii´) If for $j = 2, 3, .., n-1$, for some $k \in \Lambda$, and for t, t_0 such that $0 \le t \le t_0 \le T$, $t + d_{jk}(t) > t_0$ there is an interval $I = (t - \delta, t + \delta)$ such that $\delta > 0$ and $\inf_{t' \in I} \{x_{jk}(t')\} > 0$, then node j is reachable at time t, relative to time t_0.

The new concept of reachability extends the old one to mean the ability to get flow out of node 1 no later than some time t_0; this can be done not only by having that flow enter node n but also by virtue of it being stored at an intermediate node at time t_0 or being shipped en route at a link at that time.

In the lemma below we give sufficient conditions for a flow to be a universal maximal one. In the proof we shall need the following definition: for a given $t_0 \in [0, T]$, $v_M(t_0)$ is the value of the maximal flow on $[0, t_0]$.

Lemma 4: Suppose for a feasible flow in the network Ω that node n is not reachable relative to any time $t_0 \in [0, T]$. Then for that flow $v(t) = v_M(t)$ for all $t \in [0, T]$, i.e., it is a universal maximal flow.

Proof: Since node n is not reachable relative to time T it is a maximal flow for $t = T$. Assume there is a $t_0 \in [0, T]$ for which $v(t_0) < v_M(t_0)$. Then, according to Theorem 1 node n is reachable for time t_0 (i.e., by applying the original rules (i)-(vii) and considering the interval $[0, t_0]$ instead of $[0, T]$). If the reachability of node n is achieved by applying rules (i)-(v) only, then it is reachable also for time T, which is a contradiction. Thus, rules (vi) or (vii) must be applied to achieve reachability. If rule (vi) was applied then for some node $j \in \{2,3,\ldots,n-1\}$ $y_j(t_0) > 0$ and it follows that (for $[0,t_0]$) node n is reachable by applying rules (ii)-(vi) starting with node j at time t_0. Thus, node n is reachable relative to time t_0 contradicting the assumption. If rule (vii) was applied then there is a $j \in \{2,3,\ldots,n-1\}$ and a $t \le t_0$ such that $t + d_{jk}(t) > t_0$, for which there is some $I = (t-\delta, t+\delta)$ such that $\delta > 0$ and $\inf_{t' \in I} \{x_{jk}(t')\} > 0$. This means that, for $[0,t_0]$, node n is reachable by applying rules (ii)-(v) and (vii), starting with node j at time t. Thus, node n is reachable relative to time t_0, contradicting the assumption. □

B. Construction

Using the above lemma, we now describe a method for constructing a universal maximal flow.

Suppose that, for some feasible flow, node n is reachable relative to time t_0. It is easy to verify that there is an open interval I to the right of t_0 such that if $t \in I$ then node n is reachable relative to time t. This means that the domain $[0,T]$ can be partitioned into a countable number of open or semi-open intervals, so that for each such interval node n is either reachable relative to all times in that interval or else it is not reachable relative to any time during that interval.

Consider then a feasible flow and a node $j \neq 1,n$ such that n is reachable relative to all $t \in I = (t_1,t_2)$ or to all $t \in I = [t_1,t_2)$, for some t_1,t_2 $0 \le t_1 < t_2 \le T$, through the application of the reachability rules starting with node j. We shall say that n is reachable *from j*. Using the methods of Lemma 2, one can increase the flow such that n is no longer reachable from j relative to any t in the interval I; we shall refer to this process as *clearing* (j,I). This process makes more flow enter the destination earlier, and thus we approach the universal maximal flow monotonically.

The process of clearing (j,I) as described above may make node n reachable from more nodes and relative to new times. However, we now show that this influence is restricted. We consider the union of disjoint intervals $\bigcup \tilde{I}_i \subseteq [0,T]$, such that node n is reachable from some node relative to and only to $t \in \bigcup \tilde{I}_i$. Consider the process of clearing (j,I), as described above. Obviously, $I \subseteq \tilde{I}_l$ for some l. We show now that the clearing process may make node n reachable from new nodes but only for times t such that $t \in \tilde{I}_l$, that is, times that belong to the same interval to which I belongs.

Lemma 5: Suppose that node n is reachable from some node k relative to time t_0' after clearing (j,I), but not before. Then $t_0' \in \tilde{I}_l$, where l is such that $I \subseteq \tilde{I}_l$.

Proof: Let I, j and t_0' be as defined in the lemma. It follows that after the change of flow there is some chain L_k of node-time pairs such that all times in it are not greater than t_0', and through which flow can be sent from k to n. Since the augmentation of flow through L_k is possible only after the change of flow caused by clearing of (j,I), it follows that before that change there was a chain L_j, from node j to node n, whose times are not greater than some $t_0 \in I$, such that flow was sent through L_j, and such that there is at least one common node to L_k and L_j. We denote by r the first such node. The change of flow performed through L_j changed the flow in r (or in one of its adjacent links) in an interval I_r (whose times are not greater than t_0). The augmentation through L_k, which became possible after the change of flow, changes the flow in node r (or in an adjacent link) in the reverse manner (i.e., if L_j caused the flow to decrease then L_k increases it and vice-versa), in an interval I'_r. Obviously, $I''_r \stackrel{\Delta}{=} I'_r \cap I_r \neq \emptyset$. It is easy to verify that the times in L_j and L_k can be changed so that each is (still) a proper chain of node-time pairs, and such that the corresponding times for node r are both in I''_r. Denote by L_{jr} the prefix of L_j i.e., the portion from the beginning of L_j up to node r (more precisely - up to the appearance of r corresponding to the time in I_r). Let t_1 be the maximal time corresponding to L_{jr}. We distinguish between three cases, corresponding to the relations among t_0, t_0' and t_1.

The first case is when $t_0 \le t_0'$. Denote by L_{kr} the prefix of L_k, and by L_{rn} the suffix of L_j (where "prefixes" and suffixes" are defined in the above sense). Define L'_k as the concatenation of L_{kr} and L_{rn}. Since $t_0 \le t_0'$, all times in L'_k are not greater than t_0'. Thus, before the change of flow one could send flow from k to n through L'_k so that it arrived no later than t_0'; this contradicts the assumption that n was not reachable from k relative to t_0' before the change of flow.

The second case is when $t_1 \le t_0' < t_0$. Denote by L'_{rn} the suffix of L_k. Define L'_j as the concatenation of L_{jr} and L'_{rn}. The contradiction follows as for the first case.

The third case is when $t_0' < t_1$. Choose a time $\tilde{t}$ such that $t_0' \le \tilde{t} < t_1$. Since: (1) the time that corresponds to r in L_{jr} is in I''_r, (2) $I''_r \subset I'_r$, (3) the times in I'_r are not greater than t_0', (4) $\tilde{t} \ge t_0'$, and (5) in L_{jr} there appear times greater than t_0' (since $t_1 > t_0'$), it follows that in L_{jr} there are at least two consecutive node-time pairs with decreasing times, say: $(j_1, \tau_1), (j_2, \tau_2)$, such that $\tau_2 \le \tilde{t} \le \tau_1$; we focus our attention on the last such two consecutive pairs. If $j_1 = j_2$ then before the change $y_{j_2}(\tilde{t}) > 0$ (since $\tilde{t} \in [\tau_2, \tau_1]$); otherwise, it must be that $\tau_2 + d_{j_2 j_1}(\tau_2) = \tau_1$ and there was an interval I_2 around τ_2 such that for all $\tau \in I_2$: $x_{j_2 j_1}(\tau) > 0$. In the first case, rule (vi') held (before the change of flow) at j_2 at time $\tilde{t}$ and relative to $\tilde{t}$; in the second case, rule (vii') held at j_2 at time τ_2 and relative to $\tilde{t}$. Consider now the suffix $L_{j_2 r}$ of L_{jr} that starts with (j_2, τ_2). Since (j_1, τ_1), (j_2, τ_2) are the last two consecutive pairs that "cross" time $\tilde{t}$, and since $L_{j_2 r}$ ends at a time $t' \le \tilde{t}$, then it must be that all times in $L_{j_2 r}$ are not greater than $\tilde{t}$. As before, denote by L'_{rn} the suffix of L_k from r to n, and define $L_{j_2} \stackrel{\Delta}{=} L_{j_2 r} \cup L'_{rn}$. Since times in $L_{j_2 r}$ are not greater than $\tilde{t}$, it is easy to verify that the times of L_{j_2} can be changed so that L_{j_2} is a proper chain of node-time pairs, whose times are not greater than $\tilde{t}$. Thus, before the change of flow we could have increased flow from j_2 to n relative to time $\tilde{t}$, through the chain L_{j_2}.

Thus, we have that before the change of flow there was some node from which n was reachable relative to time $\tilde{t}$. Since $\tilde{t}$ was chosen arbitrarily from $[t_0', t_1)$, we conclude that before the change node n was reachable relative to all times in that interval. In other words, t_0' belongs to the same interval $\tilde{I}_l$ to which I belonged. □

Thus, we see that when clearing flows in some interval $\tilde{I}_l$ we do not affect times out of that interval. This property is useful from a practical standpoint. It shows that one can clear flow from all intervals of $\bigcup \tilde{I}_i$ in parallel, since there is no interference between the clearing processes of different intervals. An alternative approach for constructing universal maximal flows could be to consider successively finer discretizations and to solve the corresponding (regular) maximal flow problems. Nonetheless, in this approach there is an inherent sequentiality, since one cannot solve for a sub-interval before solving for prior ones.

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